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NOTEBOOK

Romney

Cairo, Egypt

Academic Year 1991

Ain Shams University

The Faculty of Science

The Department of Mathematics

Third Academic Year for

Mathematics Major Students

A Standard Course, and Lecture Notes in

Analytical Mechanics

The Course Presented by

Ph.D Professor Dr., and Head of the Mathematics Department

Ibrahim Fahmy Mikhail

in

Lagrangian and Hamiltonian Mechanics

with Applications to

Classical, Statistical, Relativistic and Quantum Mechanics

Presenting the Hamilton Equations for Holonomic Conservative Systems and other problems with solutions

And the Theory of Small Oscillations about the Position of Stable Equilibrium

+ Example3: Given $\underline{q} = \left(\frac{kx}{x^2+y^2}, \frac{ky}{x^2+y^2} \right)$, Show that the motion is irrotational and the fluid is incompressible, Find ϕ (potential velocity)

(1) $\nabla \times \underline{q} = 0$ (2) $\underline{q} = -\nabla \phi$

$$u = -\frac{\partial \phi}{\partial x}, \quad v = -\frac{\partial \phi}{\partial y}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

hence $\nabla^2 \phi = 0 \Rightarrow$ Laplace Equation.

+ 4 $\underline{q} = k^2 (x\,\underline{j} - y\,\underline{i}) / (x^2 + y^2)$

5 $\underline{q} = \left(-\frac{2xyz}{(x^2+y^2)^2}, \frac{(x^2-y^2)z}{(x^2+y^2)^2}, \frac{y}{(x^2+y^2)} \right)$

Analytical Mech. Dr/ Ebrahem Fahmy Ebrahem.

+ Motion of a System of particles or a rigid body in Space.

1 Forces acting as external + internal

2 + Linear motion $\dot{\underline{p}} = \underline{F}$ (I) , $\sum_i m_i \dot{\underline{r}}_i = M \ddot{\underline{r}} = \dot{\underline{p}}$

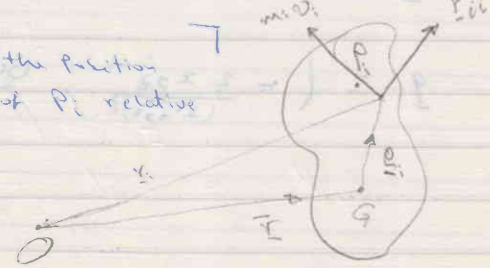
3 + Rotational motion about a fixed point O ⇒ this is taken as origin.

$\dot{\underline{h}}_o = \underline{N}_o$ (II) = angular momentum = moment of linear momentum about O = moment of external forces about O

$= \sum \underline{r}_i \times m_i \dot{\underline{r}}_i = \underline{v}_i$

4 Rotational motion about the Centeroid of gravity G.

$\underline{r}_i = \bar{\underline{r}} + \underline{\rho}_i$; $\underline{\rho}_i$ is the Position vector of P_i relative to G



$\underline{h}_o = \sum_i \underline{r}_i \times m_i \dot{\underline{r}}_i$

$= \sum_i \underline{\rho}_i + \bar{\underline{r}} \wedge m_i \dot{\underline{r}}_i = \sum \underline{\rho}_i \times m_i \dot{\underline{r}}_i + \bar{\underline{r}} \times \sum m_i \dot{\underline{r}}_i =$

Linear momentum

$\underline{h}_o = \underline{h}_G + \bar{\underline{r}} \times M \dot{\bar{\underline{r}}}$ (III) V.Imp for "Impulse"

Angular momentum about a fixed point O = A.mom about G + A.mom. of a mass M at G about O.

$\underline{N}_o = \sum_i \underline{r}_i \times \underline{F}_{ii} = \sum_i (\underline{\rho}_i + \bar{\underline{r}}) \times \underline{F}_{ii} =$

$\sum_i \underline{\rho}_i \times \underline{F}_{ii} + \bar{\underline{r}} \times \sum_i \underline{F}_{ii} = \underline{N}_G + \bar{\underline{r}} \times M \ddot{\bar{\underline{r}}}$ (4) →

No impulse because $\underline{h}_G$ is easier to get than $\underline{h}_o$

$\bar{r}$ bar

$$\dot{\bar{h}}_O = \dot{\bar{h}}_G + \bar{r} \times M \ddot{\bar{r}} + \dot{\bar{r}} \times M \dot{\bar{r}} \quad (8)$$

from (7), (8), II

$$\Rightarrow \quad \dot{\bar{h}}_G = \bar{N}_G \qquad \text{IV}$$

rate of change of angular mom about the center of gravity G = moment of external forces about G.

i.e the Center of gravity G can be treated as a fixed Point for the rotational motion.

Moreover, from eqt. (7), IV we find

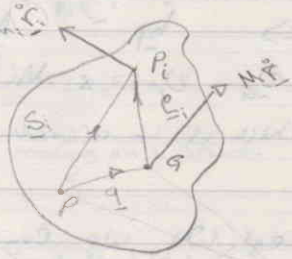
$$\bar{N}_O = \dot{\bar{h}}_G + \bar{r} \times M \ddot{\bar{r}} \rightarrow f_G \text{ acceleration} \quad (\text{V})$$

= rate of change of A. mom. about G added to M total mass times $M\ddot{\bar{r}}$ moment of the acceleration of G about ($\bar{r}$) O.

5. Rotational motion about any point (not necessarily fixed) P.

$\bar{s}_i$: Position vector of P_i relative w.r to P.

q " " " of G " " P.



$$\bar{s}_i = \bar{q} + \bar{\rho}_i$$

and A. mom. about P

$$\bar{h}_P = \sum_i \bar{s}_i \times m_i \dot{\bar{r}}_i = \sum_i (\bar{\rho}_i + \bar{q}) \times m_i \dot{\bar{r}}_i =$$

$$\sum_i \bar{\rho}_i \times m_i \dot{\bar{r}}_i + \bar{q} \times \sum_i m_i \dot{\bar{r}}_i \quad \Rightarrow$$

$$\bar{h}_P = \bar{h}_G + \bar{q} \times M \dot{\bar{r}} \quad \text{VI'}$$

which is a generalization of (VI)

mom. of Forces (external) about P =

$$\underline{N}_P = \sum_i \underline{S}_i \times \underline{F}_{ii} = \sum (\underline{\rho}_i + \underline{q}) \times \underline{F}_{ii} =$$

$$\sum \underline{\rho}_i \times \underline{F}_{ii} + \underline{q} \times \sum_i \underline{F}_{ii} = \boxed{F = M\ddot{\underline{r}}} \; \text{I}$$

$$\Rightarrow N_G = \dot{h}_G \; \text{IV}$$

Imp

$$\underline{N}_P = \dot{\underline{h}}_G + \underline{q} \times M\ddot{\underline{r}} \quad (\ddot{\underline{r}} \rightarrow \text{acceleration} \rightarrow \underline{h}_G) \quad (\text{V}') \quad \text{Generalization of eqt. IV}$$

= M. mom. of external forces about any point P =

rate of change of A. mom. about G + moment of the acceleration of G about P.

+ taking the der.

Moreover from eqt. (V') we have $\dot{\underline{h}}_P = \dot{\underline{h}}_G + \dot{\underline{q}} \times M\ddot{\underline{r}} + \underline{q} \times M\dddot{\underline{r}}$

$\dot{\underline{q}}$ = velocity of G relative to P.

$\Rightarrow$ from V'

$$\Rightarrow \dot{\underline{h}}_P = \underline{N}_P + \dot{\underline{q}} \times M\dot{\underline{r}} =$$

$$(\underline{v}_G - \underline{v}_P) \times M\dot{\underline{r}}\,\underline{v}_G = \underline{N}_P - \underline{v}_P \times M\underline{v}_G \quad (*)$$

which is in accordance with II, IV

From eqt. (*) we conclude that $\dot{\underline{h}}_P = \underline{N}_P$ in the following cases

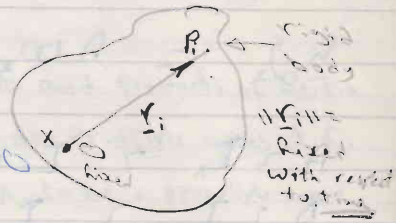
(i) P is fixed $\underline{v}_P = 0 \Rightarrow$ eqt. (II)

(ii) P coincides with with G $\Rightarrow \underline{v}_P = \underline{v}_G \Rightarrow$ (eqt. IV)

(iii) $\underline{v}_P \parallel \underline{v}_G$ (iv) $\underline{v}_G = 0$ Pure rotational motion about G.

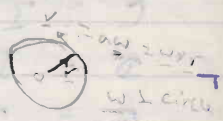
6. The angular momentum of a rigid body, fixed in one of its points O, about this point.

$$\underline{h}_O = \sum_i \underline{r}_i \times m_i \dot{\underline{r}}_i \quad \underline{v}_i$$



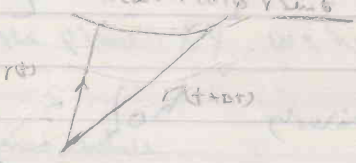
Let $\underline{\omega}$ be the angular velocity of the body

$\Rightarrow$ any point of the body P_i rotates (moves) about O on the surface of a sphere of center O, radius $\|\underline{r}_i\|$ and angular velocity $\underline{\omega}$



$\|\underline{r}\|$ = a radius of the sphere

$\Delta \underline{r} = \underline{r}(t+\Delta t) - \underline{r}(t)$



$|\Delta \underline{r}| = \omega \Delta t \, r \sin\theta$

in a direction of $\underline{\omega} \times \underline{r}$, therefore $\|\underline{v}_P\| = \lim_{\Delta t \to 0} \left| \frac{\Delta \underline{r}}{\Delta t} \right| = \omega r \sin\theta$

also the direction of $\underline{\omega} \times \underline{r}$, therefore

$\underline{v}_P = \underline{\omega} \times \underline{r}$ the velocity of a point moving on a sphere.

Therefore $\underline{v}_i = \underline{\omega} \times \underline{r}_i \Rightarrow \underline{h}_O = \sum_i \underline{r}_i \times m_i (\underline{\omega} \times \underline{r}_i)$

$= \sum_i m_i (\underline{r}_i \times (\underline{\omega} \times \underline{r}_i)) = \sum m_i \left[(\underline{r}_i \cdot \underline{r}_i)\, \underline{\omega} - (\underline{r}_i \cdot \underline{\omega})\, \underline{r}_i \right] =$

$\underline{h}_O = \sum m_i r_i^2 \underline{\omega} - m_i (\underline{\omega} \cdot \underline{r}_i)\, \underline{r}_i =$

$\sum_i m_i \left\{ (x_i^2 + y_i^2 + z_i^2)\, \underline{\omega} - (x_i \omega_x + y_i \omega_y + z_i \omega_z)\, \underline{r}_i \right\}$

$\underline{h}_{O_x}$

$\underline{h}_{O_y} \Rightarrow \underline{h}_{O_j} = \sum_i m_i x_j \underline{\omega} - \sum (x_{jk} \omega_{x_j})\, \underline{r}_i$

$\underline{h}_{O_z}$

$$\underline{h_O}_z = \omega_x \sum_i m_i (y_i^2 + z_i^2) - \omega_y \sum_i m_i y_i x_i - \omega_z \sum_i m_i x_i z_i$$

$$I_x, I_y, I_z, I_{yz}, I_{zx}, I_{xy} \Rightarrow$$

$$\underline{h_O}_x = A\omega_x - F\omega_y - E\omega_z$$

$$A = I_x = \sum m(y^2 + z^2)$$
$$B = I_y = \sum m(z^2 + x^2)$$
$$C = I_z = \sum m(x^2 + y^2)$$
$$D = I_{yz} = \sum myz$$
$$E = I_{zx} = \sum mzx$$
$$F = I_{xy} = \sum mxy$$

Similarly $h_O)_x = \omega_x A - \omega_y F - \omega_z E$

$$h_O)_y = \omega_x F + \omega_y B - \omega_z D \quad \text{VII}$$

$$h_O)_z = -\omega_x E - \omega_y D + \omega_z C$$

* Since in plane $h_O = I\omega = I\dot{\theta}$ $\Rightarrow$ VII is a Generalization

$\rightarrow$ O is a fixed point of a rigid body.

Column vector

$$\begin{bmatrix} h_O)_x \\ h_O)_y \\ h_O)_z \end{bmatrix} = \begin{bmatrix} A & -F & -E \\ -F & B & -D \\ -E & -D & C \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \quad \text{VII}$$

Column vector

matrix is called Inertia matrix at the point O

* which is a tensor of a 2nd order because matrices in general are tensors of 2nd order. $\bar{h}_O = \bar{\bar{I}}_O \bar{\omega}$

C. Mech. ; A. Dyna.

Remarks:

(i) The following notations are used for moments and products of Inertia

(ii) If Ox, Oy, OZ are chosen to be the principle axes at O then $I_{xy} = 0$, $I_{zx} = 0$ for Ox to be a principle axes

2) in general Ox, Oy, OZ be the principle axes at O when products of inertia are zeros.

Accordingly $\underline{h}_O = A\omega_x \underline{i} + B\omega_y \underline{j} + C\omega_z \underline{k}$

(iii) For a plane laminar moving in the plane XY we have

$\Rightarrow \omega_x = \omega_y = 0$ also

OZ we be a principle axis and hence

hence $\sum m_i yz$ always $= 0$

$\Rightarrow D = E = 0$

- Consequently $h_{O)x} = h_{O)y} = 0 \Rightarrow \underline{h}_{O)z} = C\omega_z = C\underline{\omega}$

Similar to the well know relation $h_o = I\dot{\theta}$

+ where I is the moment of Inertia about an axis (OZ) $\perp$ to the plane of motion and passing through the fixed point O.

(iv) For the general motion of a rigid body if O a fixed point,

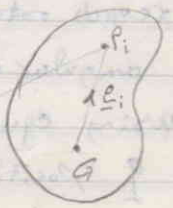
we have $\underline{h}_G = \sum_i \underline{e}_i \times m_i \dot{\underline{r}}_i =$

$\sum \underline{e}_i \times m_i (\dot{\underline{e}}_i + \dot{\underline{r}}) =$

$\sum_i m_i (\underline{e}_i \times \dot{\underline{e}}_i) + \sum_i m_i \underline{r}_i + (\sum m_i \underline{e}_i) \times \dot{\underline{r}}$

moment of the masses

0



But from the laws of the Center of gravity

$$\sum_i m_i \underline{\rho}_i = 0 \quad , \quad \sum_i m_i \dot{\underline{\rho}}_i = 0 \quad , \quad \sum m_i \ddot{\underline{\rho}}_i = 0 \qquad (1.10)$$

$$\underline{h}_G = \sum m_i \rho_i \times \dot{\rho}_i \qquad (1.11)$$

but $\dot{\underline{\rho}}_i$ represents the velocity of P_i relative to G

in this relative motion G is considered to be fixed and thus P_i moves relative to G on a surface of a sphere or with center G, radius $\|\underline{\rho}_i\|$ which is fixed in mag. and angular velocity $\underline{\omega}$ of the whole body.

Consequently $\underline{\omega} \times \underline{\rho}_i = \dot{\underline{\rho}}_i$

$$\underline{h}_G = \sum m_i \rho_i \times (\omega \times \rho_i) = \sum m_i [\rho_i^2 \underline{\omega} - (\underline{\omega} \cdot \rho_i) \rho_i]$$

$$\underline{\rho}_i = x_i' \underline{i} + y_i' \underline{j} + z_i' \underline{k}$$

Therefore

$$\begin{bmatrix} h_{G_x} \\ h_{G_y} \\ h_{G_z} \end{bmatrix} = \begin{bmatrix} A' & -F' & -E' \\ -F' & B' & -D' \\ -E' & -D' & C' \end{bmatrix} \cdot \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \qquad \text{VII}$$

$$\underline{h}_G = \bar{\bar{I}}_G \underline{\omega} \qquad \text{VIII}$$

As a regrds rotational motion G can be treated as a fixed point.

The angular momentum about any point P can be obtained by using equation (VII') i.e $\underline{h}_P = \underline{h}_G + \underline{q} \times M \dot{\underline{r}} \, v_G$

$\underline{q}$ position of G relative to P.

taking $\underline{q} = \underline{r} \Rightarrow \underline{v}_G = \underline{\omega} \times \underline{r}$

— as regard a fixed Point O, eqt. III can be used this and the result can be expressed in the same form as eqt. (V) by taking $\underline{v}_G = \underline{\omega} \times \underline{r}$ and using the theorem parallel axes.

* The Kinetic Energy of a rigid body in which one of its points is fixed i.e one of its Points is fixed

* Kinetic Energy $= T = \frac{1}{2} I_0 \dot{\theta}^2$

K.E $= T = \frac{1}{2} \sum_i m_i \dot{\underline{r}}_i \cdot \dot{\underline{r}}_i$ — $\underline{v}_i \cdot \underline{v}_i$

If O is the fixed Point of the body then $\underline{v}_i = \dot{\underline{r}}_i = \underline{\omega} \times \underline{r}_i$

Therefore $T = \sum_i m_i \dot{\underline{r}}_i \cdot (\underline{\omega} \times \underline{r}_i) =$

$\frac{1}{2} \underline{\omega} \cdot \sum_i \underline{r}_i \times m_i \dot{\underline{r}}_i = \frac{1}{2} \underline{\omega} \cdot \underline{h}_0 =$

$= \frac{1}{2} [\omega_x h_{ox} + \omega_y h_{oy} + \omega_z h_{oz}]$ => from II =

$\frac{1}{2} [\omega_x^2 A + \omega_y^2 B + \omega_z^2 C - 2\omega_x \omega_y F - 2\omega_x \omega_z E - 2\omega_y \omega_z D]$ III

$T = \frac{1}{2} \bar{\omega}^T \underline{h}_0 = \frac{1}{2} \bar{\omega}^T \bar{\bar{I}}_0 \bar{\omega}$ (transpose)

$= \frac{1}{2} \begin{pmatrix} \omega_x & \omega_y & \omega_z \end{pmatrix} \begin{bmatrix} A & -F & -E \\ -F & B & -D \\ -E & -D & C \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$

Since $T = \frac{1}{2} I \omega^2$

$M L^2 T^{-2}$

- Notes :

(i) if Ox, Oy, Oz are chosen to be the principal axes at O, then all products of inertia i.e. $D = E = F = 0$ and hence

$$T = \frac{1}{2}\left[\omega_x^2 A + \omega_y^2 B + \omega_z^2 C\right]$$

— and we always desire to take the principal axes.

(ii) For a plane ~~Lamiaer~~ laminer moving in the plane xy we have $\omega_x = \omega_y = 0$ and $D = E = 0$ $\Rightarrow$ in III

$$T = \frac{1}{2}\left[\omega_z^2 C\right] = \frac{1}{2} C\omega_z^2 = \frac{1}{2} C\|\underline{\omega}\|^2$$

in agreement with the well known result $= \frac{1}{2} I \omega^2 = \frac{1}{2} I \dot{\theta}^2$.

(iii) For the general motion of a rigid body

$$T = \frac{1}{2} \sum m_i \dot{\underline{r}}_i \cdot \dot{\underline{r}}_i =$$

$$\frac{1}{2} \sum_i m_i (\dot{\underline{c}}_i + \dot{\underline{r}}) \cdot (\dot{\underline{c}}_i + \dot{\underline{r}}) =$$

$$\frac{1}{2} \sum_i m_i \dot{\underline{c}}_i \cdot \dot{\underline{c}}_i + \frac{1}{2} \dot{\underline{r}} \cdot \sum_i m_i \dot{\underline{c}}_i +$$

$$\frac{1}{2} \dot{\underline{r}} \cdot \dot{\underline{r}} \sum_i m_i$$

Since $\dot{\underline{c}}_i = \underline{\omega} \times \underline{c}_i$ $\Rightarrow$

$$T = \frac{1}{2} \underline{\omega} \cdot \underline{h}_G + \frac{1}{2} M v_G^2 =$$

$$\frac{1}{2} \underline{\omega}^T \bar{\bar{I}}_G \underline{\omega} + \frac{1}{2} M v_G^2$$

E_K due to rotational motion about G, and due to the linear motion due to G

IX

8. Some general remarks.

(i) If rotating axes are used Then for any vector $\underline{V}$ (not velocity)

$$\left.\frac{d\underline{V}}{dt}\right|_a = \left.\frac{\delta \underline{V}}{\delta t}\right|_r + \underline{\Omega} \times \underline{V}$$

(2) (1)

($\underline{\Omega}$ (angular velocity of the axes))

hence if $\underline{h}_O = \underline{N}$

(differentiation, assuming that the axes are fixed)

Electro Magnetism.

Basic Electrostatic Conseptions and Laws:

Coulombs law

There are two kinds of interactions of elementary particles which are noticable even at large distances, electrical forces,

(a) proportional to the charge

(b) Magnetic forces which are proportional to the product of the charge and the velocity of the particle.

* In electrostatics only forces of the 1st kind exist.

- Two particles with the electric charges Q_1, Q_2 at a distance r from each other interact with a force equal to $K\, Q_1 Q_2 / r^2$
- where K is a constant of proportionality. In order to take into consideration also the direction of the force the distance vector in the direction from Q_1 to Q_2 is designated by $\underline{r}$. Then the force which acts on the charge Q_2 will be expressed by $\underline{F} = \frac{1}{4\pi\epsilon_0} \frac{Q_1 Q_2}{r^2} \hat{r}$ both in mag. and direction

This relation implies that there is repulsion and attraction if the charges have like signs and unlike signs, respectively

- the Const. of proportionality K depends on the units used. Put $K = 1/4\pi\epsilon_0$ => ϵ_0 becomes in certain natural Constant which acts as the Constant of the system of units

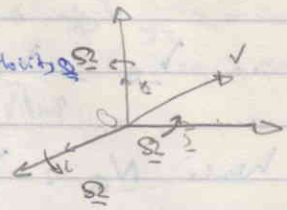
Potential

- Electric field Strength & potential; Eqt. of force and work:

Consider a number of charges $q_1, q_2, \ldots$ at the fixed points (ξ_i, η_i, ζ_i); $i = 1, 2, \ldots$ and a movable charge with the Coordinate (x, y, z) the force $\underline{F}$ acting on Q is then the vector sum of all the forces originating from the separate Q_i's

A. Mech.

Let x, y, z be rotating F. of ref. with angular velocity $\underline{\Omega}$



$$d\underline{V}/dt|_a = \frac{\partial \underline{V}}{\partial t}|_r + \underline{\Omega} \times \underline{V}$$

$\underline{\Omega}$ not necessarily constant

$$\text{then } \underline{V} = V_1 \underline{i} + V_2 \underline{j} + V_3 \underline{k}$$

$$\Rightarrow \frac{d\underline{V}}{dt} = \left(\frac{dV_1}{dt}\underline{i} + \frac{dV_2}{dt}\underline{j} + \frac{dV_3}{dt}\underline{k}\right)_r + \left(V_1 \frac{d\underline{i}}{dt} + V_2 \, d\underline{j}/dt + V_3 \frac{d\underline{k}}{dt}\right)$$

- The unit vector $\underline{i}$ can be regarded as the position vector of a point moving on the surface of a sphere of center O ; fixed point , and radius equal to unity , with an an. velocity $\underline{\Omega}$, therefore

$\frac{d\underline{i}}{dt}$ represent the velocity of this point and is thus given by

$$d\underline{i}/dt = \underline{\Omega} \times \underline{i} \qquad \text{Similarly}$$
$$d\underline{j}/dt = \underline{\Omega} \times \underline{j}$$
$$d\underline{k}/dt = \underline{\Omega} \times \underline{k} \qquad \Rightarrow$$

$$\frac{d\underline{V}}{dt} = \frac{d\underline{v}}{dt}|_r + \underline{\Omega} \times \underline{V}$$

+ The laws of motion obtained before must then be modified to include the terms which arise due to the rotation of the axes when performing any differentiation operation

- Consequently . in linear motion $\underline{\dot{p}} = \underline{F}$

(1) $M \ddot{\underline{r}}_G = \underline{F}_G$

(2) $\dot{\underline{h}}_O = \underline{N}_O$ becomes

$\dot{\underline{h}}_O + \underline{\Omega} \times \underline{h}_O = \underline{N}_O$ (2) (in rotating axes)

$$\frac{d\underline{h}_O}{dt}|_a = \frac{d\underline{h}_O}{dt}|_r + \underline{\Omega} \times \underline{h}_O$$

- also in $\underline{h}_O = \underline{h}_G + \underline{\bar{r}} \times M \underline{\dot{\bar{r}}}_O \underline{v}_G$ III

- hence $\underline{N}_G = \dot{\underline{h}}_G + \underline{\Omega} \times \underline{h}_G$ II

- $\underline{N}_P = \dot{\underline{h}}_G + \underline{q} \times M \ddot{\underline{\bar{r}}}$

$\underline{N}_O = \dot{\underline{h}}_G + \underline{\Omega} \times \underline{h}_G + \underline{\bar{r}} \times M \underline{h}_G$ II

the accelerations whose components are taken relative to the rotation axes

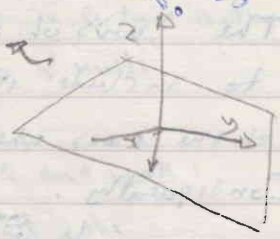
$\underline{N}_P = \dot{\underline{h}}_G + \underline{\Omega} \times \underline{h}_G + \underline{q} \times M \underline{h}_G$ I'

angular velocity of the body $\Rightarrow$ angular velocity of the axes.

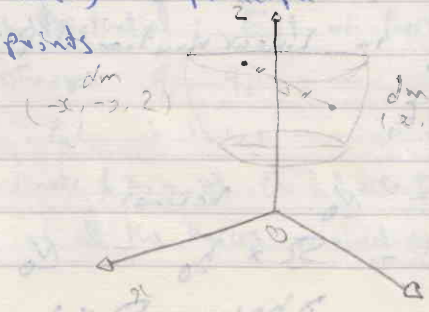
Remark (2) Some Rules to determine the principle axes of Inertia.

(a) The axis perpendicular to a plane laminar; in any state of motion, is a principle axis of inertia at the point of intersection with the plane of the laminar.

$I_{yz} = \sum m y z_{=0} = 0$; $I_{zx} = \sum m z x_{=0} = 0$



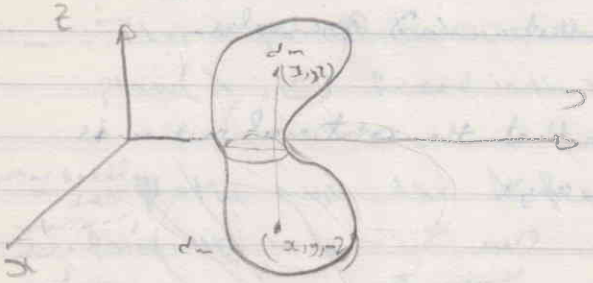
(b) The axis of symmetry is a principle axis at all of its points



$dI_{yz} = dm\, yz + dm(-y)z = 0 \Rightarrow I_{yz} = 0$

Similarly $I_{zx} = 0$; V.Imp

(c) The axis ⊥ to a plane of symmetry is a principal axis at the point of intersection with the plane of symmetry.



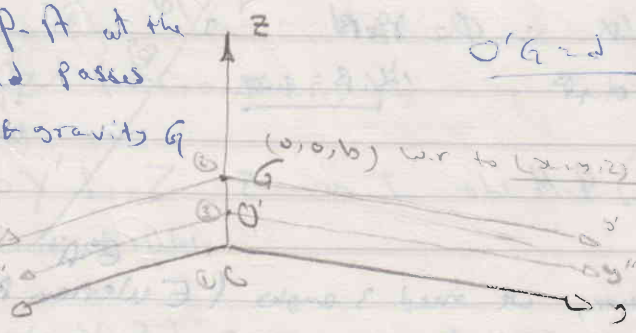
$dI_{yz} = dm\, yz + dm(-y)z = 0$

There for all z is a principal axis at the point O

P A

(d) The principal axis which passes through the Center of gravity is a Principal axis at all of its points

Let Oz be a p.a at the point O and passes through the Center of gravity G



O'G = d & OG = b

(0,0,b) w.r to (x,y,z) first reference

$I_{y'z} = I_{yz} -$

$M(0 \alpha b) =$

i.e. from O → G

From parallel axes theorem

$I_{zx'} = I_{zx} - M(b,0) = 0$

$\therefore$

$I_{y'z} = I_{y'z} + M(0\alpha d) = I_{yz} = 0$; $I_{zx'} = 0$

$G \to O'$

$\Rightarrow$ Oz is a princip. axis at any point O'

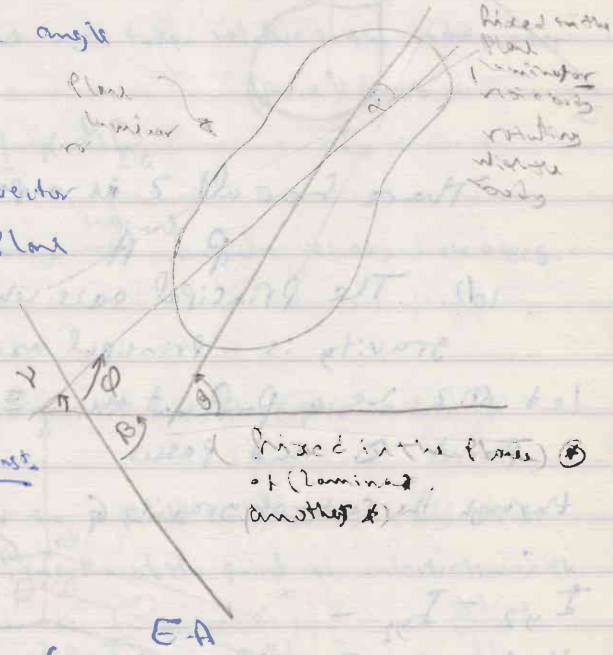
The inertia matrix $\begin{bmatrix} A & -F & -E \\ -F & B & -D \\ -E & -D & C \end{bmatrix} = M$

to get the eigen values $\Rightarrow [M - \lambda I] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix}$

$[x_1 \; x_2 \; x_3]^t$ is the eigen vector $\Rightarrow$ eigen value.

(iii) The It is well known that the rotational motion is determined in a plane by one angle

$\underline{\omega} = \dot{\theta}\,\underline{K}$ is the A. Velocity of the Laminar, $\underline{K}$ is a unit vector along the 2 direction to the Plane of motion



$\theta = \gamma + \alpha \;;\; \phi = \gamma + \beta$

$\phi = \theta - \alpha + \beta$ $\quad \alpha, \beta = \text{Const.}$

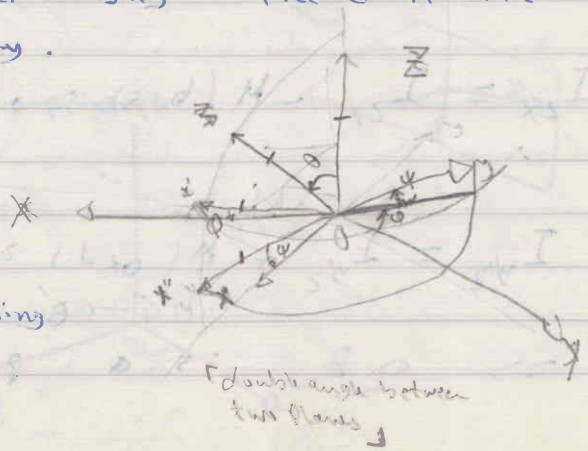
$\dot{\phi} = \dot{\theta}$

E.A

In 3. Dim, however, we need 3 angles (Eulerian Angles) to determine the angular velocity of the body. The E.A are defined in the following way.

Let X, y, Z be fixed axes in the space

Let x, y, z be 3 axes which are fixed in the body and rotating with it



- The E.A are the angle ⊖ between the axis Z (fixed in the body) and Z (fixed in the space) v.imp

- ⊖ Z is the meridian plane (in magnetic field) e.g. plane (Z, Z) is usually called the meridian plane.

- The angle ϕ between the meridian plane (Z, Z) and the plane (X, Z) fixed in the space.

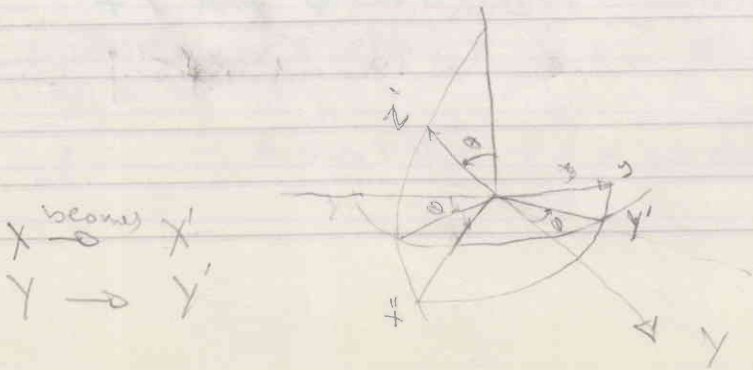
- ψ is the angle the M.Plane (Z, Z) and the plane (Z, x) which is fixed in the body

- plane ⊥ Z Contains the axes (X, Y, X')
  plane ⊥ z " " " (x, y, X'')
  ⊖ X' is the intersection of the M.P and plane ⊥ Z } intersect
  ⊖ X'' " " " M.P " ⊥ z
  plane ⊥ z (x, y, X'), Y'

Therefore ⊖ Y' ⊥ Z, z ⊥ the M.P ((Z, z), X', X'') sometimes called the Line of Nodes'

- The axes (X, Y, Z) Can be rotated to the axes (x, y, z) in the following way.

(a) Rotation about ϕ Z by the angle ϕ



fixed become rotating
(and vice versa)

(b) Rotation about Y' by an angle θ

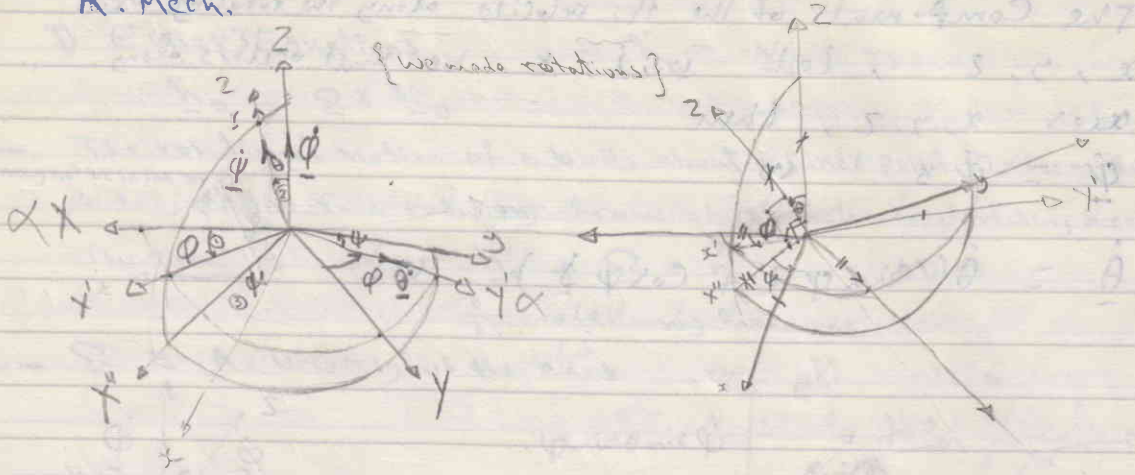
rotation in the M.P , according to this rotation A. to this R

$Z \xrightarrow{\text{becomes}} z$, $X' \xrightarrow{\text{becomes}} X''$

(c) Rotation about z by the angle ψ, (A. to this R)

$X'' \rightarrow X$, $Y' \rightarrow Y$

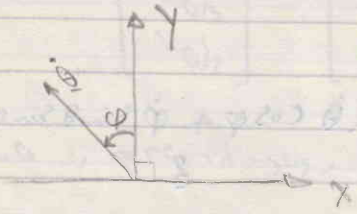
A. Mech.



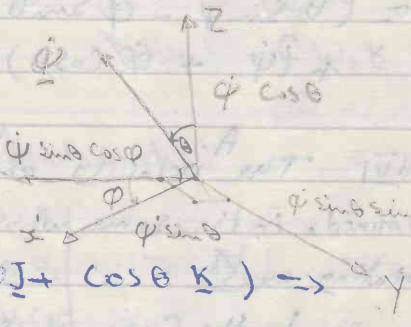
The angular velocity of the body $\underline{\omega} = \dot{\underline{\theta}} + \dot{\underline{\varphi}} + \dot{\underline{\psi}}$

+ The Components of the A. velocity along the axes X, Y, Z fixed in the space.

Let $\underline{I}$, $\underline{J}$, $\underline{K}$ be unit vectors along the fixed axes X, Y, Z, then $\dot{\underline{\varphi}} = \dot{\varphi}\,\underline{K}$ (1)



$$\dot{\underline{\theta}} = \dot{\theta}(-\sin\varphi\,\underline{I} + \cos\varphi\,\underline{J}) \quad (2)$$



$$\dot{\underline{\psi}} = \dot{\psi}\sin\theta\cos\varphi\,\underline{I} + \dot{\psi}\sin\theta\sin\varphi\,\underline{J} + \dot{\psi}\cos\theta\,\underline{K} \quad (3) \Rightarrow$$

$$\dot{\underline{\psi}} = \dot{\psi}(\sin\theta\cos\varphi\,\underline{I} + \sin\theta\sin\varphi\,\underline{J} + \cos\theta\,\underline{K}) \Rightarrow$$

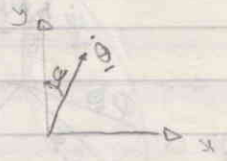
$$\underline{\omega} = (-\dot{\theta}\sin\varphi + \dot{\psi}\sin\theta\cos\varphi)\,\underline{I} + (\dot{\theta}\cos\varphi + \dot{\psi}\sin\theta\sin\varphi)\,\underline{J} + (\dot{\varphi} + \dot{\psi}\cos\theta)\,\underline{K}$$

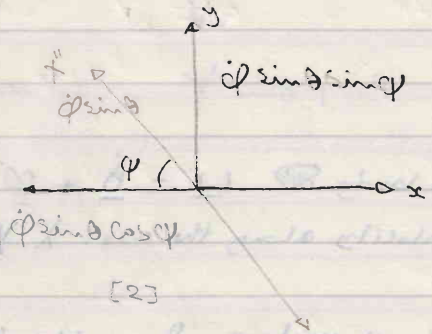
1

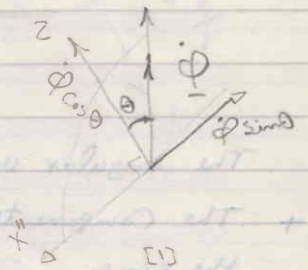
The Components of the A. velocity along the rotating axes x, y, z, let $\underline{i}, \underline{j}, \underline{k}$ be unit vectors along the axes x, y, z, then

$$\underline{\dot{\varphi}} = \dot{\varphi}\,\underline{k} \qquad (1)$$

$$\underline{\dot{\theta}} = \dot{\theta}(\sin\varphi\,\underline{i} + \cos\varphi\,\underline{j}) \qquad (2)$$







$$\Rightarrow \quad \underline{\dot{\varphi}} = \dot{\varphi}(-\sin\theta\cos\varphi\,\underline{i} + \sin\theta\sin\varphi\,\underline{j} + \cos\theta\,\underline{k})$$

$\Rightarrow$ therefore

$$\underline{\omega} = (\dot{\theta}\sin\varphi - \dot{\varphi}\sin\theta\cos\varphi)\,\underline{i} + (\dot{\theta}\cos\varphi + \dot{\varphi}\sin\theta\sin\varphi)\,\underline{j} + (\dot{\varphi} + \dot{\varphi}\cos\theta)\,\underline{k} \qquad \underline{2}$$

Note [1] The A. velocity of the rotating ~~axes~~ can also be determined, in the same way, save that (x, y, z) represent now the rotating axes.

* For example: In the case of spherical polar Coordinates: the axes (x, y, z) are taken to be (x'', y', z)

where here $\varphi = 0 \Rightarrow \dot{\varphi} = 0$ ①

hence $\underline{\dot{\theta}} = \dot{\theta}\,\underline{j}$ ② , $\underline{\dot{\varphi}} = -\dot{\varphi}(\sin\theta)\,\underline{i} + \dot{\varphi}\cos\theta\,\underline{k}$ ③

$$\Rightarrow \quad \underline{\omega} = -\dot{\varphi}\sin\theta\,\underline{i} + \dot{\theta}\,\underline{j} + \dot{\varphi}\cos\theta\,\underline{k} \qquad \underline{3}$$

Applications :

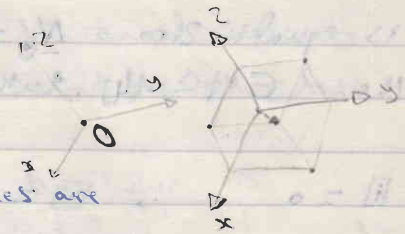
1) Euler's Equations : $\dot{\underline{h}}_G = \underline{N}_G \Rightarrow$

$\dot{\underline{h}}_O + \underline{\Omega} \times \underline{h}_O$

- The rotational motion of a body about a fixed point O (for any fixed point, (it may not be in the body), relative to a rotating frame of reference, or, rotating axes, is given by $\underline{N}_O = \dot{\underline{h}}_O + \underline{\Omega} \times \underline{h}_O$ (1)

$\frac{\partial}{\partial t}$ (assuming fixed axes)

- $\underline{\Omega}$ = A. velocity of the axes



- If we further assume that the rotating axes are fixed in the body, then $\underline{\Omega} = \underline{\omega}$ = A. velocity of the body (2)

+ Moreover, it will be assumed that O is a fixed point of the body, $\begin{bmatrix} h_{O)_x} \\ h_{O)_y} \\ h_{O)_z} \end{bmatrix} = \begin{bmatrix} A & -F & E \\ -F & B & -D \\ E & -D & C \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$ and the rotating axes are the principal axes of inertia of the body at the point O fixed

I_x I_y I_z

Hence, accordingly, $\underline{h}_O = A\omega_1 \underline{i} + B\omega_2 \underline{j} + C\omega_3 \underline{k}$ (3)

Substituting from (2) & (3) in (1), we find that

$$A\dot{\omega}_1 \underline{i} + B\dot{\omega}_2 \underline{j} + C\dot{\omega}_3 \underline{k} + \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \omega_1 & \omega_2 & \omega_3 \\ A\omega_1 & B\omega_2 & C\omega_3 \end{vmatrix} = \underline{N}_O$$

(i)

$$A\dot{\omega}_1 - \omega_2\omega_3(B-C) = N_{O)_x} \quad \text{(ii)} \quad ; \quad B\dot{\omega}_2 - \omega_3\omega_1(C-A) = N_{O)_y} \quad \text{(iii)}$$

$$C\dot{\omega}_3 - \omega_1\omega_2(A-B) = N_{O)_z} \quad \text{(iii)} \quad (4)$$

Condition for E. eqt.s to hold:

1. O is a fixed point of the body
2. The rotating axes are along the principal axes of inertia of the body at O.

Hence, Equations (4) are called Euler's Equations.

- If we further assume that, the moment of external forces about O is equal to zero $\underline{N}_O = 0$ or the body moves under no forces, then Eqt.s (4) take the form

$$\text{(i)} = 0 \quad , \quad \text{(ii)} = 0 \quad , \quad \text{(iii)} = 0 \qquad (5)$$

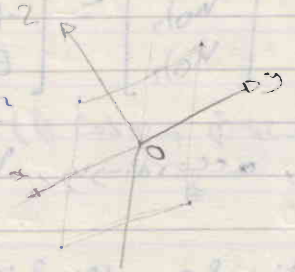
- Note: For a plane laminar, if the z axis is taken perpendicular to the plane of the laminar, then x, y are the other two principal axes at the point O in the plane of the lamina =>

$$I_z = I_x + I_y =$$

$$\sum m(y^2 + z^2) = \sum m y^2 + \sum m x^2$$

$$C = A + B \quad =>$$

=> in (4)



$$\text{(i)} = 0 = \dot{\omega}_1 + \omega_2 \omega_3 = 0 \quad (1)$$

$$\text{(ii)} = 0 = \dot{\omega}_2 - \omega_3 \omega_1 = 0 \quad (2)$$

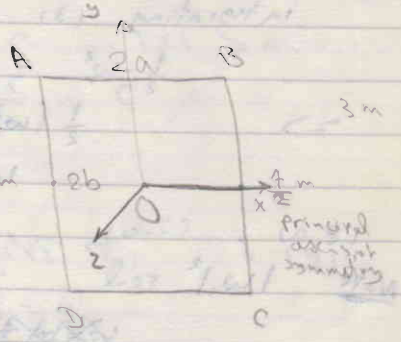
taking from (1), (2) => $\omega_3 = \dot{\omega}_2 / \omega_1$ =>

$\omega_1 \dot{\omega}_1 + \omega_2 \dot{\omega}_2 = 0$ => integrating w.r to t

$\frac{1}{2}(\omega_1^2 + \omega_2^2) = \text{Const.}$ =>

$\omega_1^2 + \omega_2^2 =$ Constant, For a plane lamina, which is determined from initial conditions

+ Example: A, B, C, D is a rectangular plane lamina of mass 3m, and in which BC = 2AB, two particles each of mass $\frac{1}{2}m$ are fixed at the mid points of DA and CB, the lamina is free to rotate about its center O which is fixed; the rectangular axes Ox, y, z are chosen parallel to AB, DA and ⊥ to the plane of the lamina, initially the lamina is set in motion with an A. velocity the Components of which are $(\sqrt{3}\Omega, 0, \Omega)$ Prove that after time t, the Component of the angular velocity along Oy is $\sqrt{3}\Omega \tanh t\Omega$ and find the other two Components.



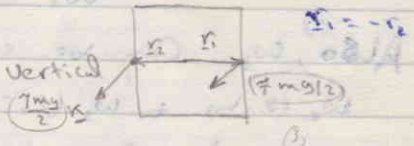
- we 1st note that

1 the center O of the laminar is fixed.

2 Ox, Oy are symmetrical axes passing through O, and are thus principal axes of inertia at O, also Oz ⊥ Ox, Oy i.e. to the plane of the lamina, and is accordingly a principal axis.

3 $N_O = \underline{r}_1 \times \frac{7mg}{2}\underline{k} + \underline{r}_2 \times \frac{7mg}{2}\underline{k} = 0$

⇒ we can use eqn. (5).



+ We can thus apply Euler's eqns in the form

$A\dot{\omega}_1 - \omega_2\omega_3(B-C) = 0$ (1.a)

$B\dot{\omega}_2 - \omega_3\omega_1(C-A) = 0$ (1.b)

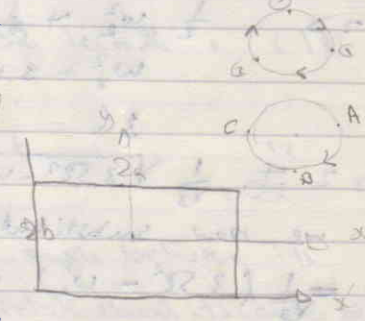
$C\dot{\omega}_3 - \omega_1\omega_2(A-B) = 0$ (1.c)

Let $AB = 2a \Rightarrow BC = 2AB = 4a$

Moment of inertia of a rectangle $= \frac{1}{3}Mb^2$

$A = I_x = \frac{1}{3}\times 3m\times(2a)^2 + 0 + 0 = 4ma^2$

$B = I_y = \frac{1}{3}\times 3m\times a^2 + \frac{1}{2}ma^2 + \frac{1}{2}ma^2 = 8ma^2$



For a plane lamina $C = I_z = I_x + I_y$

therefore $A : B : C = 4ma^2 : 8ma^2 : 12ma^2 = 1 : 2 : 3$

Substituting in equation (1.a,b,c) we find

→ $\dot{\omega}_1 - \omega_2\omega_3(2-3) = 0 \Rightarrow \dot{\omega}_1 + \omega_2\omega_3 = 0$ (2.a) (for any plane lamina is a general one) [also prove it]

→ $2(\dot{\omega}_2) - \omega_3\omega_1(3-1) = 0 \Rightarrow 2\dot{\omega}_2 - 2\omega_3\omega_1 = 0 \Rightarrow$

$\dot{\omega}_2 - \omega_3\omega_1 = 0$ (2.b)

→ $3\dot{\omega}_3 - \omega_1\omega_2(1-2) = 0 \Rightarrow 3\dot{\omega}_3 + \omega_1\omega_2 = 0$ (2.c)

initially we have 3 components at $t=0$, we have 2 components

– In order to express ω_1 interm of ω_2 we use the following way.

$\omega_1 \times (2.a) + \omega_2 \times (2.b) = \omega_1\dot{\omega}_1 + \omega_2\dot{\omega}_2 = 0$ (3)

integrating (3) w.r to t →

$\frac{1}{2}\omega_1^2 = -\frac{1}{2}\omega_2^2 + C_1$ (4), at $t=0$ $\omega_2 = 0$ (From the initial conditions) $\Rightarrow C_1 = \frac{3}{2}\Omega^2$

$\Rightarrow \frac{1}{2}(\omega_1^2 + \omega_2^2) = \frac{3}{2}\Omega^2 \Rightarrow$

$\omega_1^2 + \omega_2^2 = 3\Omega^2 = (\sqrt{3}\Omega)^2$ (4)

(2.a), (2.b), (4) are correct for any plane lamina

Note $|\omega_1|^2$ real $\Rightarrow |\omega_2| < (\sqrt{3}\Omega) \Rightarrow$

$\omega_2 = \pm\sqrt{3\Omega^2 - \omega_1^2}$ (5)

we take the (+ve) sign since initially $\omega_2 = 0$, $\omega_1 = +\sqrt{3}\Omega$

$\omega_1 = \sqrt{3\Omega^2 - \omega_2^2}$ (5), also

→ **Also**, ω_3 can be expressed interms of ω_2 as follows

$\omega_2(2.b) + \omega_3(2.c) = \omega_2\dot{\omega}_2 + 3\omega_3\dot{\omega}_3 = 0$ (6)

$\Rightarrow \frac{1}{2}\omega_2^2 + \frac{3}{2}\omega_3^2 = C_2 \Rightarrow C_2 = \frac{3}{2}\Omega^2$ {From initial conditions}

$\omega_2^2 + 3\omega_3^2 = 3\Omega^2 \Rightarrow$

$3\omega_3^2 = 3\Omega^2 - \omega_2^2$ $|\omega_2| < \sqrt{3}\Omega$

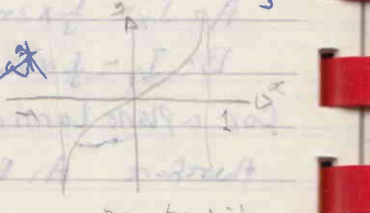
$\omega_3 = \pm\frac{1}{\sqrt{3}}\sqrt{3\Omega^2 - \omega_2^2}$ (7)

→ we now substitute (5), (7) in (2.b) we find that

$\dot{\omega}_2 = \frac{1}{\sqrt{3}}(3\Omega^2 - \omega_2^2) = \frac{d\omega_2}{dt}$ (8) $\Rightarrow \int \frac{dt}{\sqrt{3}} = \int \frac{d\omega_2}{3\Omega^2 - \omega_2^2}$

$x = \tanh y = \frac{e^{2y} - 1}{e^{2y} + 1}$

$y = \tanh^{-1} x = \frac{1}{2}\ln\frac{1+x}{1-x}$



$\Rightarrow$ A. Mech.

$$\int \frac{1}{2}\frac{dx}{1-x^2} = \frac{1}{2}\left(\frac{1}{1-x}+\frac{1}{1+x}\right)dx = \frac{1}{2}\ln\frac{1+x}{1-x} \qquad |x|<1$$

Notice: 1. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C = -\frac{1}{a}\cot^{-1}\frac{x}{a} + C'$

2. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tanh^{-1}\frac{x}{a} + C = \frac{1}{2a}\ln\frac{a+x}{a-x} + C \quad |x|<a$

3. $\int \frac{dx}{x^2-a^2} = \frac{1}{a}\coth^{-1}\frac{x}{a} + C \quad |x|>a$

$= -\frac{1}{2a}\ln\frac{x+a}{x-a} + C$.

$t + C_3$, $|\omega_2| < \sqrt{3}\,\Omega$

$\Rightarrow \frac{t}{\sqrt{3}} + C_3 = 1/(\sqrt{3}\,\Omega)\ \tanh^{-1}(\omega_2/(\sqrt{3}\,\Omega))$ ... (9)

From initial conditions $\Rightarrow C_3 = 0$

$\Rightarrow \Omega t = \tanh^{-1}[\omega_2/(\sqrt{3}\,\Omega)]$ (9)

$\Rightarrow \omega_2 = \sqrt{3}\,\Omega\tanh(\Omega t)$ (9)

$\rightarrow$ Substituting in Eqt. (5) (4) $\rightarrow$

$$\omega_1 = -\sqrt{3}\,\Omega\sqrt{1-\tanh^2\Omega t} = \sqrt{3}\,\Omega\,\mathrm{sech}^2\Omega t \quad (10)$$

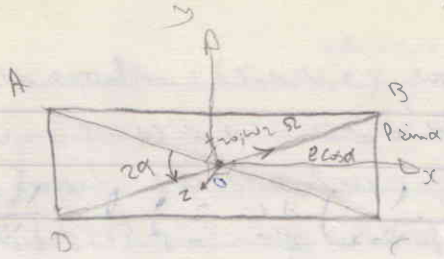
$$\omega_3 = \frac{1}{\sqrt{3}}\sqrt{3\Omega^2-\omega_2^2} = \Omega\,\mathrm{sech}\,\Omega t \quad (12)$$

+

A uniform rectangular lamina, the acute angle between its diagonals is (2α), the lamina is free to rotate about its center which is fixed; initially it started spinning about one of its diagonals with Angular velocity Ω - show that it will be spinning entirely about the other diagonal after a time

$$t = \frac{2}{\Omega\sqrt{\cos 2\alpha}}\int_0^{\pi/2}\frac{dx}{\sqrt{1-\sin^2\alpha\sin^2 x}}$$

[which is an elliptical integral solved by tables of Gauss interpolation method according to $\sin^2\alpha$]



The Center O is a fixed point of the Lamina

we also choose the axes in the following way Ox along DC (Symmetrical axis) and therefore it is a Principal axis about O)

Oy along DA . (always choose right turned system)

+ OZ ⊥ to the plane of the Lamina at O => Principal axis at O.

+ Also the moment of external forces about O is equal to zero.

Accordingly we can use Euler's equations in the form

$$A\dot{\omega}_1 - \omega_2\omega_3(B-C) = 0 \quad (1)$$

$$B\dot{\omega}_2 - \omega_3\omega_1(C-A) = 0 \quad (2)$$

$$C\dot{\omega}_3 - \omega_1\omega_2(A-B) = 0 \quad (3)$$

$DB = 2\ell$, $OB = \ell$

$A = I_x = \frac{1}{3} m \ell^2 \sin^2\alpha$ (a)

$B = I_y = \frac{1}{3} m \ell^2 \cos^2\alpha$ (b)

$C = I_z = I_x + I_y = \frac{1}{3}\ell^2$ (c)

$A : B : C = \sin^2\alpha : \cos^2\alpha : 1$

substituting in 1 , 2 , 3 =>

$$\sin^2\alpha\,\dot{\omega}_1 - \omega_2\omega_3(\cos^2\alpha - 1) = 0 \quad \sin^2\alpha(\dot{\omega}_1 - \omega_2\omega_3) = 0 \quad (2.a)$$

$$\dot{\omega}_1 - \omega_2\omega_3 = 0 \quad (2.a)$$

$$\cos^2\alpha\,\dot{\omega}_2 - \omega_3\omega_1(1 - \sin^2\alpha) = 0 \Rightarrow$$

$$\dot{\omega}_2 - \omega_2\omega_3 = 0 \quad (2.b)$$

$$\dot{\omega}_3 - \omega_1\omega_2(\sin^2\alpha - \cos^2\alpha) = 0 \Rightarrow$$

$$\sin^2\alpha - \cos^2\alpha = -\cos 2\alpha$$

$$\dot{\omega}_3 + \cos 2\alpha\,\omega_1\omega_2 = 0 \quad (2.c)$$

at t=0 $\underline{\omega} = (\Omega\cos\alpha , \Omega\sin\alpha , 0)$, $\Omega = \omega$ along DB

$\|\underline{\omega}\| = \Omega = (\omega_1^2 + \omega_2^2 + \omega_3^2)^{1/2}$

=> The target: the component of the A. velocity in the plane of the Lamina $(\omega_1^2 + \omega_2^2)^{1/2}$ can be determined in the following way

$\omega_1 (2.a) + \omega_2 \times (2.b) \Rightarrow \omega_1 \dot{\omega}_1 + \omega_2 \dot{\omega}_2 = 0$

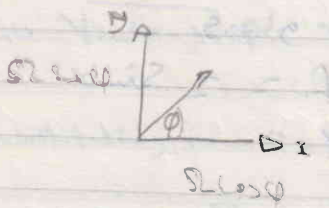
integrating w.r.to t we obtain

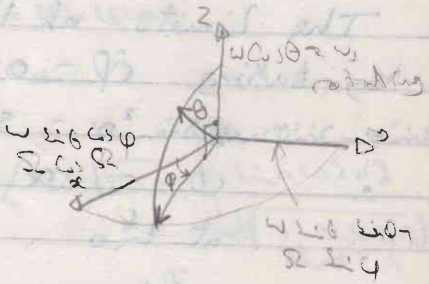
$\omega_1^2 + \omega_2^2 = C_1 = \Omega^2 \quad (5)$

$C_1 = \Omega^2 \Rightarrow [\omega_1^2 + \omega_2^2]^{1/2} = [\Omega^2]^{1/2} = \Omega \quad (5)'$

Eqts (5), (5)' indicate that the component of the A. velocity in the plane of the lamina $\sqrt{\omega_1^2 + \omega_2^2}$ remains const. at any time and is equal to Ω.

+ Let ϕ be the angle between this component and x axis at any time t





Therefore $\omega_1 = \Omega \cos\phi$

$\dot{\omega}_1 = -\Omega \sin\phi \, \dot{\phi} = -\Omega \sin\phi \, \phi'$

$\omega_2 = \Omega \sin\phi \; ; \; \dot{\omega}_2 = \Omega \cos\phi \, \phi' \quad (6)$

- Substitution of (6) in (2.a) leads to

$-\Omega \sin\phi \, \phi' + \Omega \sin\phi \, \omega_3 = 0$

$[\omega_3 = \phi'] \quad (7)$

at $t = 0$, $\omega_3 = 0 \Rightarrow \phi' = 0 \Rightarrow$ initially $\phi = \alpha \quad (8)$ ← initial conditions

- Substitution in (2.b) yields the same result.

Finally, substitution of (6) in (2.c) gives

$\dot{\omega}_3 + \Omega^2 \cos 2\alpha \cos\phi \sin\phi = 0 \quad (9)$

$\phi'' + \Omega^2 \cos 2\alpha \cos\phi \sin\phi = 0$

$\phi'' + \frac{1}{2}\Omega \cos 2\alpha \sin 2\alpha = 0 \quad (10)$

$\frac{d\phi'}{d\phi} \cdot \frac{d\phi}{dt} = \phi' \frac{d\phi'}{d\phi} \Rightarrow \int (10) \, w.r.t \, \phi$

$\frac{1}{2}\dot{\phi}^2 = -\frac{1}{2} 2\Omega \cos 2\alpha \cos 2\alpha$

$\dot{\phi}^2 = -(2\Omega \cos 2\alpha) \cos 2\alpha + C$

$$\dot{\phi}^2 + \Omega^2 \cos 2\alpha \sin^2\phi = C_2 \qquad (10)$$

from the initial conditions (9)

$\dot{\phi} = 0$ & $\phi = \alpha$ $\Rightarrow$ $C_2 = \Omega^2 \cos 2\alpha \sin^2\alpha$

$\Rightarrow$

$$\dot{\phi}^2 = \Omega^2 \cos 2\alpha \,(\sin^2\alpha - \sin^2\phi) \qquad (11)$$

From (11), of course, $\sin^2\phi \leq \sin^2\alpha$ for $\dot{\phi}^2$ to be (+ve) and $\dot{\phi}$ to be real.

i.e. $|\phi| \leq \alpha$ or $-\alpha \leq \phi \leq \alpha$ [at $t=0$]

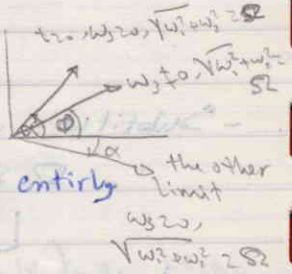
The limits of the motion are determined from (11) by taking $\dot{\phi} = 0 = \omega_3$ which gives

$\sin^2\alpha - \sin^2\phi = 0 \Rightarrow \sin^2\alpha = \sin^2\phi$ $\quad \sin\phi = \pm \sin\alpha$

$\therefore \quad \phi = \alpha$, or $\phi = -\alpha$

at $t = 0$,

$\dot{\phi} > 0$ & $\dot{\phi} = \omega_3 > 0$



the limit $\phi = -\alpha$ at $\dot{\phi} = \omega_3 = 0$ represents the position at which the plane lamina is spinning entirly about the other diagonal aC,

in order to find the required time, we integrate the integration eqn. (11) in the following way.

Since ϕ decreases with time $\Rightarrow$

$$\dot{\phi} = -\Omega \sqrt{\cos 2\alpha}\, \sqrt{\sin^2\alpha - \sin^2\phi}$$

* ϕ changes from α to $-\alpha$ i.e ϕ decreases and $\dot{\phi} = -\omega_3$

integration by parts

$$\frac{d\phi}{dt} = \rightarrow \int_0^{t=} dt' = -\frac{1}{\Omega\sqrt{\cos 2\alpha}} \int_{\alpha}^{-\alpha} \frac{d\phi}{\sqrt{\sin^2\alpha - \sin^2\phi}}$$

required time.

$$\Rightarrow t = \frac{2}{\Omega\sqrt{\cos 2\alpha}} \int_0^{\alpha} \frac{d\phi}{\sqrt{\sin^2\alpha - \sin^2\phi}}$$

even function.

A. Mech.

Changing the variables of integration $\Rightarrow$ $\int_{\sin\alpha}^{1} \frac{dx}{\sqrt{a^2-x^2}}$

$\sin\lambda = \sin\phi / \sin\alpha$

$x = a\sin\lambda$

$\sin\phi = \sin\lambda \sin\alpha$

we now use the transformation

$\therefore \phi = 0 \to \alpha \Rightarrow \lambda = 0 \to \lambda = \pi/2$ (13a)

in comparison with the the integrand we have to solve now

$\cos\phi \, d\phi = \sin\alpha \cos\lambda \, d\lambda$

$d\phi = \frac{\sin\alpha \cos\lambda \, d\lambda}{\cos\phi} = \frac{\sin\alpha \cos\lambda \, d\lambda}{\sqrt{1-\sin^2\alpha \sin^2\phi}} =$

Since $\sqrt{\sin^2\alpha - \sin^2\phi} = \sqrt{\sin^2\alpha - \sin^2\alpha \sin^2\lambda} = \sin\alpha\cos\lambda$ (13-c)

Substituting in (12) we find $t = 2/\Omega\sqrt{\cos 2\alpha} \int_0^{\pi/2} \frac{\sin\alpha\cos\lambda \, d\lambda}{\sin\alpha\cos\lambda \sqrt{1-\sin^2\alpha\sin^2\lambda}}$

$= \frac{2}{\Omega\sqrt{\cos 2\alpha}} \int_0^{\pi/2} \frac{d\lambda}{\sqrt{1-\sin^2\alpha\sin^2\lambda}}$

elleptic disc of excentricity $e = 1/\sqrt{2}$

$A = I_x = \frac{1}{4} m b^2$

$B = I_y = \frac{1}{4} m a^2$

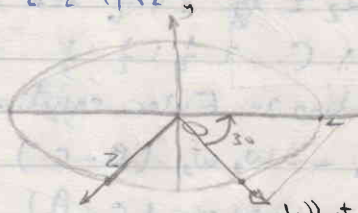
$b^2 = a^2(1-e^2)$, $e = 1/\sqrt{2}$

$b^2 = a^2/2$

$A = \frac{1}{8} m a^2$, $B = \frac{1}{4} m a^2$ (i)

$C = A + B = \frac{3}{8} m a^2$

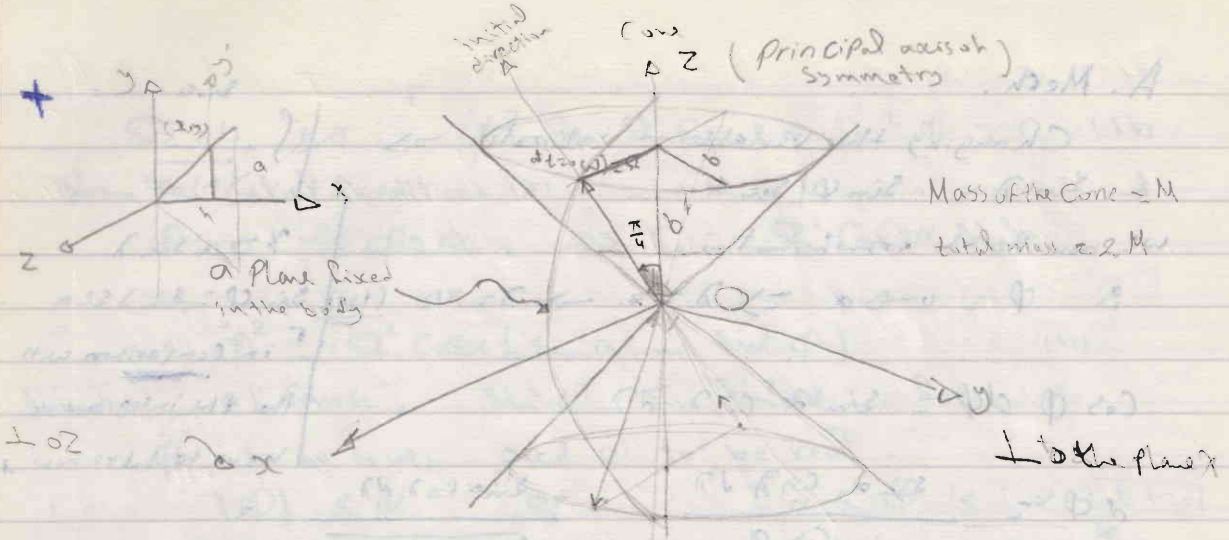
$A : B : C = 1 : 2 : 3$



ω at $t=0$ $= \Omega$

$\perp y$

$\omega_{t=0} = \Omega\left(\frac{\sqrt{3}}{2}, 0, \frac{1}{2}\right)$



$dI_x = \frac{1}{2}(\pi y^2 dx \rho)\, y^2$, $\rho = 3M/(1/3)\pi a^2 h$

$I_x = \frac{3}{10} M a^2$, $dI_y = \frac{1}{4}(\pi y^2 dx \rho)\, y^2$

$dI_y = dI_{y'} = (\pi y^2 dx \rho)\, e^2$

$I_y = 3/20\ M a^2 + \frac{3}{5} M h^2$

$A = I_x = 2\left(\frac{3}{20} M a^2 + \frac{3}{5} M h^2\right)$

$B = I_y = 2Mb^2\left(\frac{3}{20} + \frac{1}{5}\right) = \frac{3}{2} Mb^2$

$C = I_z = 2\left(\frac{3}{10} M a^2\right) = 2 M b^2 \left(\frac{3}{10}\right)$

$A : B : C = \frac{1}{2} : \frac{1}{2} : \frac{1}{5} = 5 : 5 : 2$

Substituting in Euler equations :

$5A\dot{\omega}_1 - 3\omega_2\omega_3(B-C) = 0$

$5B\dot{\omega}_2 - 3\omega_3\omega_1(C-A) = 0$

$2C\dot{\omega}_3 - \omega_1\omega_2(A-B) = 0$

$\Rightarrow \omega_3 = \text{Const.} = \Omega/\sqrt{2}$ because $\dot{\omega}_3 = 0$

$5\dot{\omega}_1 - 3\omega_2 \frac{\Omega}{\sqrt{2}} = 0$ — I

$5\dot{\omega}_2 + 3\frac{\Omega}{\sqrt{2}}\omega_1 = 0$ — II

From I $\quad 5\ddot{\omega}_1 = 3\frac{\Omega}{\sqrt{2}}\dot{\omega}_2$

$25\ddot{\omega}_1 = -\left(\frac{3\Omega}{\sqrt{2}}\right)^2 \omega_1$ $\quad$ so $\ddot{\omega}_1 = -\left(\frac{3\Omega}{5\sqrt{2}}\right)^2 \omega_1$

$= -p^2\omega_1$ S.H.M

$\omega_1 = A\cos(pt + \epsilon)$

from initial conditions $\omega_1 = A$ $\omega_1 = \frac{\Omega}{\sqrt{2}}$ / $\omega_2 = 0$ from I

$\dot{\omega}_1 = 0 \Rightarrow \omega_1 = \frac{\Omega}{\sqrt{2}} \cos(pt)$

from I $\Rightarrow \omega_2 = \frac{3\sqrt{2}}{3\Omega} \dot{\omega}_1 = \frac{1}{p}\left(-\frac{\Omega}{\sqrt{2}} p \sin pt\right)$

$\omega_2 = -\frac{\Omega}{\sqrt{2}} \sin pt$ i.e. $\omega_3 = \Omega/\sqrt{2}$

Since to return to rotate about the same generator

$\omega_2 = 0$ at $\Rightarrow pt = 0, \pi, 2\pi \Rightarrow t = 0, \frac{\pi}{p}, \frac{2\pi}{p}$

$\omega_1 = \frac{\Omega}{\sqrt{2}}, -\frac{\Omega}{\sqrt{2}}, \frac{\Omega}{\sqrt{2}}$

at $t = \frac{2\pi}{p} = \frac{6\sqrt{2}}{3\Omega}\pi \Rightarrow$

$\underline{\omega} = \left(\frac{\Omega}{\sqrt{2}}, 0, \frac{\Omega}{\sqrt{2}}\right)$ along the same generator.

another solution from I & II

$\Rightarrow$ Let (I) + i(II) $\Rightarrow$

$\dot{\omega}_1 + i\omega_1 + i\left(\frac{3\Omega}{\sqrt{2}}\right)(\omega_1 + i\omega_2) = 0$

$\Rightarrow \dot{\xi} + ip\xi = 0$ $\quad d\xi/dt = \dot{\xi}$

$\ln \xi = -ipt = \text{const.} = \ln \frac{\Omega}{\sqrt{2}}$

$\ln \frac{\sqrt{2}\xi}{\Omega} = -ipt$

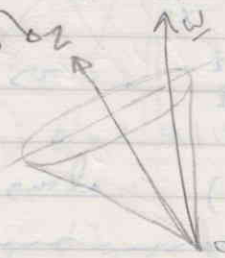
$\xi = \frac{\Omega}{\sqrt{2}} e^{-ipt}$, $\omega_1 = \frac{\Omega}{2}\cos pt$; $\omega_2 = -\frac{\Omega}{2}\sin pt$

Application II :

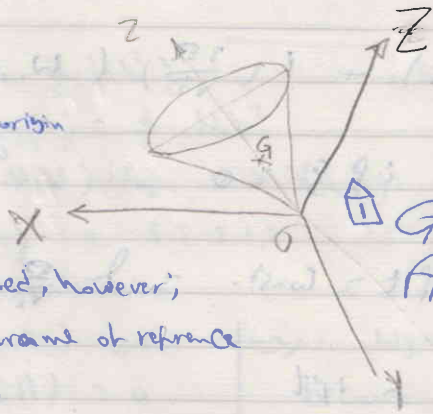
The Gyroscopic Motion :

+By the gyroscopic motion we mean the ~~rotation~~ spinning of a symmetrical rigid body about its axis of symmetry when it is mounted, so that the direction its angular velocity and axis change freely in space.

A typical example of such motion is the spinning top. (the toy does)



Example : Study the motion of a spinning symmetrical top whose apex (vertex) is fixed or at rest on a perfectly rough plane. (to prevent any sliding)



Geometry and Axes:
O Z (O III)

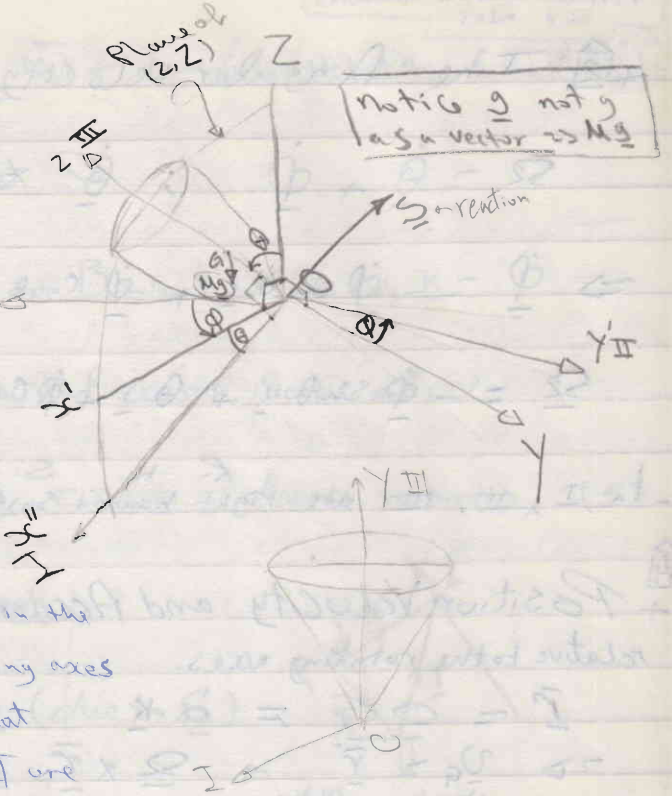
1 Geometry and Axes.

we take the fixed apex as origin
(X, Y, Z) be fixed axes in space

The motion will be studied, however, relative to a rotating frame of reference taken in the following way

i.e along the line OG where G is the center of gravity which must exist on the symmetrical axes or otherwise density isn't regular. this axis is fixed in the body.

A. Mech.



The axis OI perpendicular to the axis OIII in the meridian plane [The plane(z,Z)] along the direction in which θ increases

- The axis OII ⊥ to the meridian plane in the direction in which φ increases

- It is clear that the axes OI, , OII are not fixed in the top, they are however rotating axes

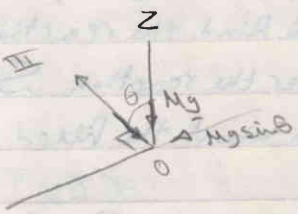
+ Moreover, it is readily shown that the three axes OIII, OI, OII are principal axes of inertia of the top at the point O [" OIII is an axes of symmetry and OI, OII are ⊥ to planes of symmetry].

(3) - External forces:

1. The force of gravity, $\underline{Mg}$ along the vertical downwards: along OZ downwards, Therefore

$$\underline{Mg} = -Mg\,\underline{K}, \quad \underline{K} \text{ s.t } \|\underline{K}\| = 1 \text{ along } OZ$$

$$= Mg\sin\theta\,\underline{i} - Mg\cos\theta\,\underline{K} \quad (1)$$



where $\underline{i}, \underline{j}, \underline{K}$ are unit vectors along the rotating axes OI, OII, OIII

2. The reaction $\underline{S}$, unknown in magnitude and direction.

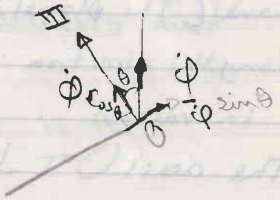
its position after Geometry and axes

+ [2] The Angular Velocity of the axes

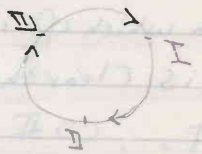
$$\Omega = \dot{\theta} + \dot{\phi} \quad ; \quad \dot{\theta} = \dot{\theta}\, j \quad ; \quad \dot{\phi} = \dot{\phi} K$$

$$\Rightarrow \dot{\phi} = -\dot{\phi}\sin\theta\, i + \dot{\phi}\cos\theta\, K \Rightarrow$$

$$\Omega = -\dot{\phi}\sin\theta\, i + \dot{\theta}\, j + \dot{\phi}\cos\theta\, K \quad (2)$$



- In I, II, III are right handed system rotating axes. R. H. S.



[4]
+ Position Velocity and Acceleration of G relative to the rotating axes.

$$\bar{r} = \overline{OG} = a K \quad , a = OG$$

$$\Rightarrow \underset{\text{absolute}}{v_G} = \underset{\text{relative}}{\dot{\bar{r}}} + \Omega \times \bar{r} \quad ; \ \Omega \text{ angular velocity of the axes.}$$

$$= \underset{\text{Zero}}{\dot{\Omega}} + \begin{vmatrix} i & j & K \\ -\dot{\phi}\sin\theta & \dot{\theta} & \dot{\phi}\cos\theta \\ 0 & 0 & a \end{vmatrix} = a\dot{\theta}\, i + a\dot{\phi}\sin\theta\, j \quad . \quad (3)$$

* The general Component of velocity in spherical polar Coordinates for a Point moving on the surface of a sphere.

$$f_G = \dot{v}_G + \Omega \times v_G$$

Should be Calculated to study the linear motion and find the reaction S.

However the reaction S Can be Cancelled by studing the rotational motion about the fixed Point O.

+ [5] The Rotational motion about the fixed point O.

$$\dot{h}_O = N_O \Rightarrow \dot{h}_O + \Omega \times h_O = N_O$$

in order to study the R. motion about the Point O we use the eqt.

$$\underset{\text{relative}}{\dot{h}_O} + \Omega \times \underset{\text{angular momentum about } O}{h_O} = N_O \text{ (moment of external forces)} \quad (4)$$

but $\underline{h}_O$ ($\bar{h}_O$ fixed point) $= \bar{\bar{I}}_O \bar{\omega}$ — angular velocity of the Top.

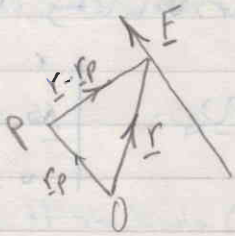
$\bar{\bar{I}}$ inertia matrix at O

Therefore $\underline{h}_O = A\omega_1 \underline{i} + B\omega_2 \underline{j} + C\omega_3 \underline{K}$

Since OI & OII are Principal axes of inertia at O.

$\Rightarrow \underline{h}_O = A\omega_1 \underline{i} + A\omega_2 \underline{j} + C\omega_3 \underline{K}$ (due to symmetry) (5)

$\underline{N}_P = (\underline{r} - \underline{r}_P) \times \underline{F}$ and here



$$\underline{N}_O = \underline{r} \times \underline{F} = \underline{0} \text{ (due to } \underline{S}) + \underline{r} \times M\underline{g} =$$

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{K} \\ 0 & 0 & a \\ Mg\sin\theta & 0 & -Mg\cos\theta \end{vmatrix} \quad (6)$$

Substituting from (5), (6) in (4) we find

$$A\dot{\omega}_1 \underline{i} + A\dot{\omega}_2 \underline{j} + C\dot{\omega}_3 \underline{K} + \begin{vmatrix} \underline{i} & \underline{j} & \underline{K} \\ -\dot{\phi}\sin\theta\,\dot{\theta} & \dot{\phi}\cos\theta & \\ A\omega_1 & A\omega_2 & C\omega_3 \end{vmatrix} =$$

$Mg\, a\,(+\sin\theta\, \underline{j}) \Rightarrow 0$

$$A\dot{\omega}_1 + C\omega_3\dot{\theta} - A\omega_2\dot{\phi}\cos\theta = 0 \quad (I)$$
$$A\dot{\omega}_2 + A\omega_1\dot{\phi}\cos\theta + C\omega_3\dot{\phi}\sin\theta = Mga\sin\theta \quad (II)$$
$$C\dot{\omega}_3 - A\omega_2\dot{\phi}\sin\theta - A\omega_1\dot{\theta} = 0 \quad (III)$$

5 unknowns ($\omega_1, \omega_2, \omega_3, \theta, \phi$)

Kinematic Condition

If description of motion with no forces involved

(6) The Kinematical Conditions (K.C)

(i.e Conditions without considering forces).

These Conditions can be obtained in one of the following ways

i) Since O is a fixed point of the top, then @ the velocity of G (as a point of the body is given by

$\underline{v}_G = \underline{\omega} \times \underline{\bar{r}}$; ($\underline{\omega}$ = angular velocity of the body.)

G moves about O, on a surface of a sphere with the angular velocity $\underline{\omega}$ of the body and radius $\bar{r}$.

$$\Rightarrow \underline{v}_G = \begin{vmatrix} i & j & k \\ \omega_1 & \omega_2 & \omega_3 \\ 0 & 0 & a \end{vmatrix} = a\omega_2 \underline{i} - a\omega_1 \underline{j} \quad (7)$$

from (3), (7) $\Rightarrow$ $\omega_2 = \dot{\theta}$ (II)

$\omega_1 = -\dot{\phi}\sin\theta = -\omega_1$ (I)

} the Kinematical Conditions

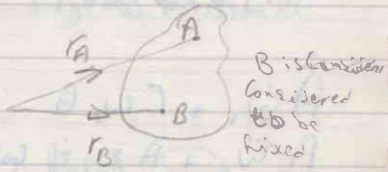
which decreases the unknowns into 3 only.

(ii) Since the plane is perfectly rough (no sliding, slipping), then

$\underline{v}_{O)_{\text{O as a point of the top}}} = \underline{v}_{O)_{\text{O as a point of the plane}}} = 0$; O fixed

from the law of relative velocity $\underline{v}_A = \underline{v}_B + \underline{v}_{A \text{ rel } B} = \underline{v}_{AB}$

$\Rightarrow \underline{v}_G + \underline{v}_{OG} = 0$



$\underline{\omega} \times (\underline{r}_A - \underline{r}_B) \Rightarrow$

$\underline{v}_G + \underline{\omega} \times (\underline{r} - \underline{r}_G) = 0$ ($\underline{r}$: (0,0,0), $\underline{r}_G$: (0,0,a))

$\underline{v}_{AB} = \underline{\omega} \times \underline{BA}$

$= \omega \times (\underline{r}_A - \underline{r}_B)$

$$\Rightarrow \quad a\dot{\theta}\,\underline{i} + a\dot{\phi}\sin\theta\,\underline{j} + \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \omega_1 & \omega_2 & \omega_3 \\ 0 & 0 & -a \end{vmatrix} = 0 \quad \Rightarrow$$

$a\dot{\theta} - a\omega_2 = 0$, leads to $\Rightarrow$ IV

$a\dot{\phi}\sin\theta - a\omega_1 = 0 \Rightarrow$ I

(iii) In order to determine the angular velocity of the top, we should rotate the axes OI, OII by the Eulerian angle ψ about the axis OIII, which is fixed in the top, accordingly OI (OX'') $\Rightarrow$ Ox (fixed in the top); OII (OY') $\Rightarrow$ Oy (fixed in the top)

Therefore $\underline{\omega} = \dot{\underline{\phi}} + \dot{\underline{\theta}} + \dot{\underline{\psi}} = \underline{\Omega} + \dot{\underline{\psi}}$

$\underline{\Omega}$ is the angular velocity of the axes.

$\dot{\underline{\psi}}$ represents the relative angular velocity of the body (Top) relative to the axes.

$\Rightarrow \underline{\omega} = \underbrace{-\dot{\phi}\sin\theta\,\underline{i} + \dot{\theta}\,\underline{j} + \dot{\phi}\cos\theta\,\underline{k}}_{\text{from (2)}} + \dot{\psi}\,\underline{k}$ ← rotation about OZ.

$\Rightarrow \omega_1 = -\dot{\phi}\sin\theta$ I, $\omega_2 = \dot{\theta}$ IV

$\omega_3 = \dot{\phi}\cos\theta + \dot{\psi}$ (8).

Solution of the Principal eqtn I, II, III, IV, I.

- We now substitute from the kinematical conditions IV, I in III (the simplest)

$\Rightarrow C\dot{\omega}_3 - A\dot{\theta}\dot{\phi}\sin\theta + A(-\dot{\phi}\sin\theta)\dot{\theta} = 0$

$\Rightarrow C\dot{\omega}_3 = 0$ i.e. $\omega_3 = \text{Constant} = \eta$

i.e. The A. velocity of the top about the axis of symmetry is Const. $= \eta$ III'

From (8) $\omega_3 = \dot{\phi}\cos\theta + \dot{\psi} = \text{Const} = \eta$

Lastly, substitution of the K.C II & I in I

gives or yields

$$A(-\ddot{\varphi}\sin\theta - \dot{\varphi}\dot{\theta}\cos\theta) + C\eta\dot{\theta} - A\dot{\theta}\dot{\varphi}\cos\theta = 0 \quad (\dot{\omega}_y)$$

$$-A(\ddot{\varphi}\sin\theta + 2\dot{\varphi}\dot{\theta}\cos\theta) + C\eta\dot{\theta} = 0$$

$$\Rightarrow -A(\ddot{\varphi}\sin^2\theta + 2\dot{\varphi}\dot{\theta}\cos\theta) + C\eta\dot{\theta}\sin\theta = 0$$

($d\dot{\varphi}/dt$; $\dot{\varphi}\sin\theta\, d\theta/dt$)

$$\Rightarrow -A\frac{d}{dt}(\dot{\varphi}^2\sin^2\theta) + C\eta\sin\theta\,\dot{\theta} = 0$$

integrating w.r to t $\Rightarrow$

$$-A\int d(\dot{\varphi}^2\sin^2\theta) + C\eta\int\sin\theta\, d\theta = \text{Constant.}$$

$$-A\dot{\varphi}\sin^2\theta + C\eta\cos\theta = \text{Const.} \Rightarrow$$

$$A\dot{\varphi}\sin^2\theta + C\eta\cos\theta = H \qquad (I)'$$

[Note if the vertical position ($\theta = 0$) is a position of the motion then $H = C\eta$]

* Alternative method to find (I') by Angular momentum.

It is clear that the moment of external forces the vertical axes $oZ = 0$ (since M_y is parallel to oZ and S intersects oZ at O).

Also the rate of change of angular momentum = the moment of external force). Accordingly the vertical Component of the Angular momentum about o h_o = Const.

Therefore

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$h_{z_0} = C\omega_3 \cos\theta + A\omega_1 \sin^2\theta = \text{Const.} = H$

integrating

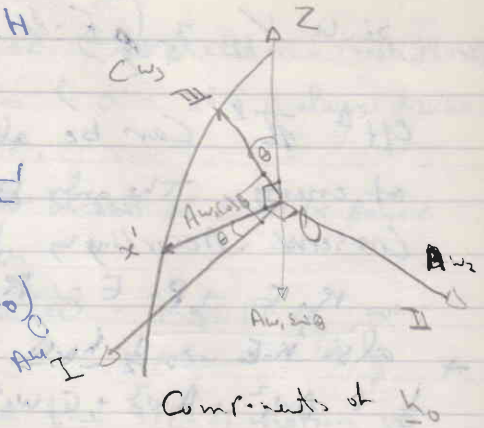
$C\eta \cos\theta - A\sin\theta(-\dot\phi \sin\theta) = H$

$\Rightarrow C\eta\cos\theta + A\dot\phi\sin^2\theta = H$ I'

H is, therefore, the vertical component of h_0 (or any one about O h_0)

Also from [I'] we have

$\dot\phi = H - C\eta\cos\theta \,/\, A\sin^2\theta$ (11)



Components of h_0

* Finally we substitute from the kinematical conditions IV, V in eqt^n. II. Therefore,

$A\ddot\theta - A\dot\phi^2\sin\theta\cos\theta + C\eta\dot\phi\sin\theta = Mga\sin\theta$ (12)

* This eqt^n will be used later to study the steady motion.

eqt^n. (12) can be integrated by substituting for $\dot\phi$ from (11) taking

$\ddot\theta = \dot\theta\frac{d\dot\theta}{d\theta}$ and integrating w.r.t. θ

+ However, it may be much easier to use eqt^n. (10) instead

According to (10) we have

$A\ddot\theta - \dot\phi\sin\theta(C\eta - A\dot\phi\cos\theta) = Mga\sin\theta$ (12)'

$\Rightarrow A(\ddot\phi\sin\theta - \dot\phi\dot\theta\cos\theta) + \dot\theta(C\eta - A\dot\phi\cos\theta) = 0$

multiplying by $\dot\phi\sin\theta$ we find

$-A(-\ddot\phi\dot\phi\sin^2\theta + \dot\phi^2\dot\theta\sin\theta\cos\theta) + \dot\theta\dot\phi\sin\theta(C\eta - A\dot\phi\cos\theta) = 0$

$\frac{A}{2}\frac{d}{d\theta}(\dot\phi^2\sin^2\theta) = \dot\phi\sin\theta(C\eta - A\dot\phi\cos\theta)$ (13)

Substituting from (13) in (12)' we find

$A\ddot\theta + \frac{A}{2}\frac{d(\dot\phi^2\sin^2\theta)}{d\theta} = Mga\sin\theta$

$A\ddot\theta d\theta$ integrating w.r.t. θ

$\frac{A}{2}(\dot\theta^2 + \dot\phi^2\sin^2\theta) = -Mga\cos\theta + \text{Const.}$

$A(\dot\theta^2 + \dot\phi^2\sin^2\theta) + 2Mga\cos\theta = \text{Const.}$ II'

Kinetic energy Pot. Energy

since $V = \int_G^e F \cdot dr = -Mg\,\overline{GZ} = \int_Z^e F \cdot dr = Mg\,a\cos\theta$

eqt^n II' can be alternatively obtained by using the Conservation of energy. The only force which performs work is Mg which is Conserved. Accordingly Kinetic energy + Potential energy = Const

$K.E + P.E = K$

also $K.E \to \frac{1}{2} I_{ow} \omega$, O is a fixed point

$\frac{1}{2}\left(A\omega_1^2 + A\omega_2^2 + C\omega_3^2 \right)$

(from eqt I')

$= \frac{1}{2}A(\dot\phi^2 \sin^2\theta + \dot\theta^2) + \frac{1}{2} C\eta^2 + Mga\cos\theta = K'$

Therefore

$\frac{A}{2}(\dot\phi \sin^2\theta + \dot\theta^2) + Mga\cos\theta = K' - \frac{C\eta^2}{2}$

eqtns I', II', III' are the 3 principle eqtns which describe the motion of a Spinning top.

→ The Condition of steady motion:

Steady motion is defined by $\theta = Const. = \alpha$,

$\dot\phi = Const. = \omega$ $\Rightarrow$ $\dot\theta = \ddot\theta = 0$; $\ddot\phi = 0$ (14)

In order to find the required conditions we substitute from (14) in (12) hence,

$-A\omega^2 \sin\alpha\cos\alpha + C\eta\omega\sin\alpha = Mga\sin\alpha$

Either, $\sin\alpha = 0$ $\alpha = 0$, π which represents the vertical position

OR: $A\omega^2\cos\alpha - C\eta\omega + Mga = 0$ (16)

Therefore $\omega = \left[\eta \pm \sqrt{C^2\eta^2 - 4Mga A\cos\alpha}\right] / (2a\cos\alpha)$ (17)

notice (16), (17) not necessarily are to invalid at the vertical position

ω is real i.e $C^2\eta^2 - 4Mga A\cos\alpha \geq 0$

which is the required condition for the steady motion is $C^2\eta^2 \geq 4Mga\cos\alpha A$

* Note (1) If the apex O is fixed (not at rest on a rough plane) The steady motion in the horizontal position ($\alpha = \frac{\pi}{2}$) is always possible since $\cos\alpha = 0$ and (18) is satisfied.
In this position, the angular velocity $\dot\phi$ in this * can be found directly from eqtn (16), according to which

$$-C\eta\dot\phi + Mga = 0 \Rightarrow \dot\phi = \frac{Mga}{C\eta} \quad (+ve)$$

* Note (2) It seems from eqtn (15) that the steady motion in the vertical position ($\theta = 0$) is possible, therefore it is of interest to study small oscillations about the vertical position $\theta = 0$.
For this motion $\theta \approx 0$ (small quantity), $\dot\theta, \ddot\theta \approx 0$, as well as $\sin\theta = \theta - \frac{\theta^3}{3!} + \text{n.t}$, $\cos\theta = 1 - \frac{\theta^2}{2} + \text{n.t}$ (19)
"n.t negligible terms."

From I', $A\dot\phi^2\sin^2\theta + C\eta\cos\theta = h$ I'
Since the vertical is a position of the motion.
Substituting from (19) we have

$$A\dot\phi^2\theta^2 + C\eta\left(1 - \frac{\theta^2}{2}\right) = C\eta$$

(θ^2: 2nd order term; 2nd order term.)

$$\Rightarrow \theta^2\left(A\dot\phi^2 - \frac{C\eta}{2}\right) = 0 \Rightarrow$$

$$\dot\phi = \frac{C\eta}{2A} = \text{Const.} \quad (20)$$

which can not be obtained from eqtn (17).
Substituting in eqtn (12), then

$$A\ddot\theta = -A\left(\frac{C\eta}{2A}\right)^2\theta \times 1 + C\eta\left(\frac{C\eta}{2A}\right)\theta - Mga\theta$$

($\ddot\theta$: small quantity)

$$A\ddot\theta = -\frac{C^2\eta^2}{4A}\theta + Mga\theta \quad \text{(for stable steady motion)}$$

$$\ddot\theta = -\left(\frac{C^2\eta^2}{4A} - Mga\right)\theta \Rightarrow$$

$$\ddot\theta = -\frac{1}{A}\left(\frac{C^2\eta^2}{4A} - Mga\right)\theta \quad (21)$$

(S.H.M)

* For a stable steady motion we must have $C^2\eta^2 > 4Mgae$ (21)

" " " " " about the vertical position

subject to Condition (21) eqtn (20) represents a S.H.Oscillation of periodic time $= 2\pi/\omega =$

$$\frac{2\pi\,(2A)}{\sqrt{C^2\eta^2 - 4Mgae}} = T \qquad (22) \qquad C^2\eta^2 \geq 4Mgae\cos\alpha$$

eqtn (21) equivalent to Condition (18) with $\alpha = 0$ (Vertical position)

+ The Limits of the General Motion:

$$\bar{z} = a\cos\theta \quad , \quad \dot{\bar{z}} = -a\sin\theta\,\dot{\theta}$$

in order to find the limits of the motion we substitute for $\dot{\phi}$ for eqtn II, in eqtn I

$$\dot{\phi} = (H - C\eta\cos\theta)/A\sin^2\theta \qquad (\text{eqtn } 4)$$

Substituting in eqtn III we find

$$\frac{A}{A}\left[\dot{\theta}^2 + \sin^2\theta\left(\frac{H - C\eta\cos\theta}{A\sin^2\theta}\right)^2\right] + \frac{2Mgae\cos\theta}{A} =$$

$$= \frac{K}{A} \quad \Rightarrow$$

$$\left[\dot{\theta}^2 + \sin^2\theta\left(\frac{H - C\eta\cos\theta}{A\sin^2\theta}\right)^2\right] + \frac{2Mgae\cos\theta}{A} = \frac{K}{A}$$

$$\dot{\theta}^2\sin^2\theta = \left(\frac{K}{A} - \frac{2Mgae}{A}\cos\theta\right)\sin^2\theta - \left(\frac{H}{A} - \frac{C\eta}{A}\cos\theta\right)^2 \qquad (23)$$

if $u = \cos\theta = \frac{\bar{z}}{a} \Rightarrow \dot{u} = \dot{\bar{z}}/a = -\sin\theta\,\dot{\theta} \Rightarrow$ (24)

Substituting in eqtn.

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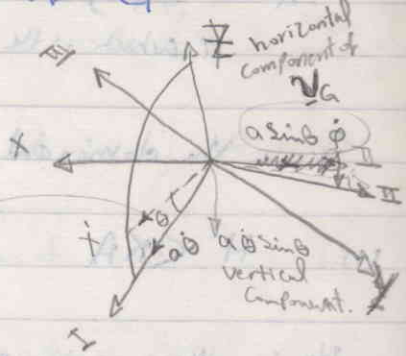
$\dot{Z}$ = vertical velocity of G.



Substituting in eqtn (23) =>

$$\dot{u}^2 = \left(\frac{K}{A} - \frac{2Mg}{A}u\right)(1-u^2) - \left(\frac{H}{A} - \frac{CN}{A}u\right)^2 \quad (25)$$

$$\dot{u}^2 = (\alpha - \beta u)(1-u^2) - (\gamma - \delta u)^2$$

The limits of the motion can be obtained from eqtn (25) by putting $\dot{\theta} = 0$ $\dot{u} = 0$. including the vertical position $\theta = 0$, which might not be a limit to the motion

In order to study the existance of these limits we proceed in the following way.

Let $\dot{u}^2 = f(u) = (\alpha - \beta u)(1-u^2) - (\gamma - \delta u)^2$

3rd degree eqtn => 3 roots

- For large u dominant term βu^3 =>

∴ Since the dominant term for large u is βu^3

θ	π	$\frac{\pi}{2}$
u	$-\infty$	-1 no root ; 0
$f(u)$	$(-ve)$	$(+ve)$ $\alpha - \gamma^2$
	$(\gamma+\delta)^2$	$\frac{KA - H^2}{A^2}$ +ve if $H^2 < KA$ -ve if $H^2 > KA$

From initial Conditions :-

at $t = 0$, $\theta = \theta_0$ => $f(u)$ (+ve)

u_0 θ => one of the limits of the motion i.e. vertical position

$f(u)$ (+ve) $-(\gamma - \delta)^2$

∴ the vertical position is a position of the motion.

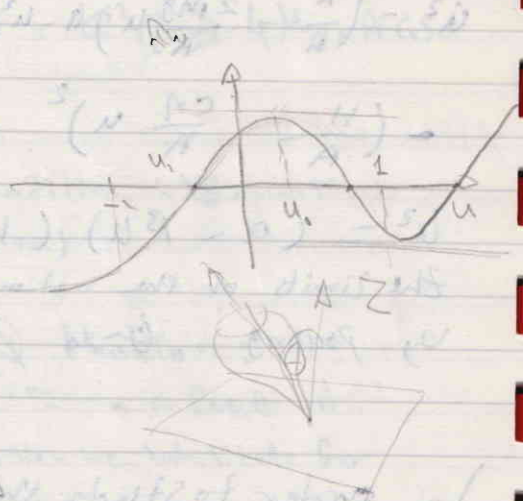
If $\dot{\theta} = 0$ for the vertical position ($\theta = \pi$) then

$K = 2mga \Rightarrow \alpha = \beta$ (double root)

(which is the situation common in most problems))

Since the dominant term is βu^2

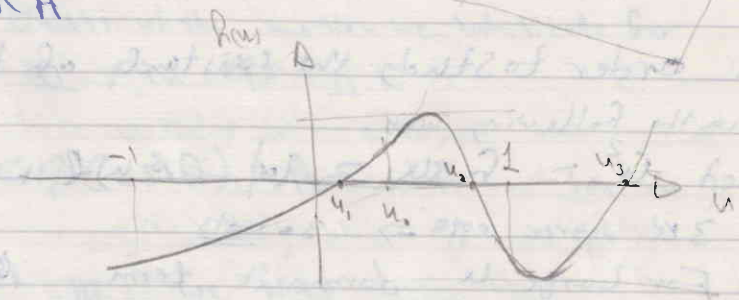
(1) $H^2 < KA$



Hence there is two possible limits for the motion.

(2) if $H^2 > KA$

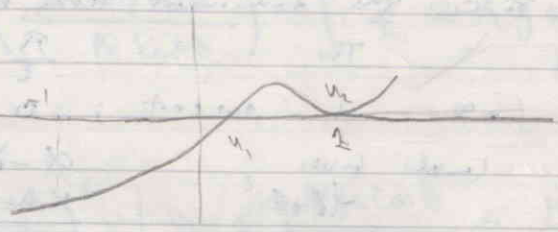
apex fixed or



(3)

$\dot{\theta} = 0$, $b \neq 0$

$H = cH$, $K = cMg$



- it is clear from the fig. above that if $H^2 > Ka$ two roots are possible (u_1, u_2).
- if $H^2 < KA$ two roots are possible (u_1, u_2) if the Apex O is fixed.
+ while there exist only one root u_2, if the apex O possible

is at rest on a perfectly rough plane

finally from eqtn (26) $\dot{u} = \pm\sqrt{f(u)} = \frac{du}{dt}$

to get the time.

but, if the top falls down. +

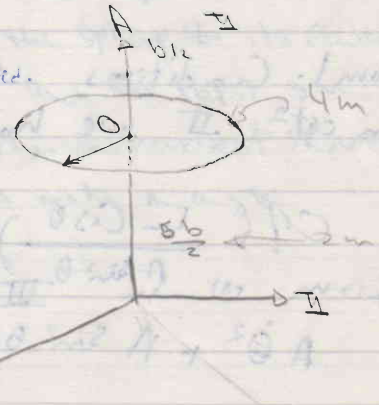
(but) if the top rises up.

$$t = \pm \int \frac{du}{\sqrt{f(u)}} + C.$$

(which gives an elliptic integral)

example (1) A Top consists of a uniform circular disc of mass 4m, and of radius b, and center O and a uniform thin rod AOB of mass 2m and length 3b the rod is perpendicular to the plane of the disc and passes through its center s.t AO = b/2 ; OB = 5b/2 initially AB is vertical and B is fixed and the top is then given a spin $5\sqrt{139/b}$ about its axis.

If the top is slightly disturbed show that its axis will fall until it will make an angle $\cos\theta = 1/3$ $\cos^{-1}(\frac{a}{b})$ with the upward vertical.



Perform steady motion till slightly disturbed.

1. The initial Conditions.

$\omega_3 = n = 5\sqrt{139/b}$,

$\underline{v}_G = \underline{0} = a\dot{\theta}\,\underline{i} + a\dot{\phi}\sin\theta\,\underline{j}$

$\dot{\theta} = 0$

2. From the geometry of the fig. A, C, ϕ a

$$I_{III} = C = \frac{1}{2}(4m)b^2 = 2mb^2$$

$$I = A = \frac{1}{4}(4m)b^2 + I'_{disc}$$

$$I_{1)_{disc}} = 4m\left(\frac{5b}{2}\right)^2 + \frac{4}{3}(2m)\left(\frac{3b}{2}\right)^2$$

(rod mass; half the length)

$$A = mb^2 + 25mb^2 + 6mb^2 = 32mb^2$$

$$M_{a_{BG}} = \underset{disc}{4m\left(\frac{5b}{2}\right)} + \underset{rod}{(2m)\left(\frac{3b}{2}\right)} = 13mb$$

3. The 3 principal eqt^n of the top are

$$\omega_3 = \eta = \text{constant} \quad (I)$$

$$A\dot{\phi}\sin^2\theta + C\eta\cos\theta = H \;;\; t \to 0,\ \theta \to 0$$

$\therefore$ Therefore $H = C\eta$ II

$$A(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + 2Mga\cos\theta = K$$

From I. Conditions $t \to 0$, $\dot{\theta} \to 0$, $\theta \to 0$, $K = 2Mg$ III

From eqt^n II we have $\dot{\phi} = \dfrac{H - C\eta\cos\theta}{A\sin^2\theta}$

$$C\eta\left(\frac{1-\cos\theta}{A\sin^2\theta}\right) \quad (1)$$

From (1) in III we have

$$A\dot{\theta}^2 + A\sin^2\theta\, C^2\eta^2(1-\cos\theta)^2/(A\sin^2\theta) +$$

$$2Mga\cos\theta = 2Mga \quad (2)$$

In order to find the limits of the motion we take $\dot{\theta} = 0$ in eqn (2), therefore,

$$\frac{C^2\eta^2}{A}(1-\cos\theta)^2 = 2Mga(1-\cos\theta)(1-\cos^2\theta)$$

$$\frac{C^2\eta^2}{A}(1-\cos\theta)^2 - 2Mga(1-\cos\theta)^2(1+\cos\theta) = 0$$

$$(1-\cos\theta)^2\left[\frac{C^2\eta^2}{A} - 2Mga(1+\cos\theta)\right] = 0$$

∴ the limits of the motion are given by

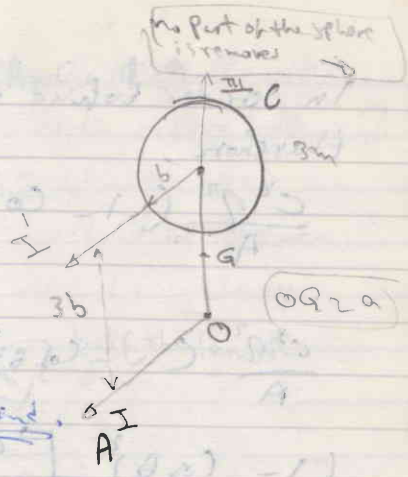
$\rightarrow$ $1-\cos\theta = 0 \Rightarrow \theta = 0$, (double root)

$\rightarrow$ or $1+\cos\theta = C^2\eta^2/2Mga A$ $= \frac{4m^2b^4 \times 25}{(2mb^2)(5mb^2)(32mb^2)}$

$= \frac{25}{16} \Rightarrow \cos\theta = \frac{25}{16} - \frac{16}{16} = \frac{9}{16}$

$\theta = \cos^{-1}\frac{9}{16}$.

* A Top Consist of a uniform solid sphere of mass $5m$ and radius b, and of a thin uniform rod of mass m and length $3b$ penetrating into the sphere to its Center the motion Started with the rod vertical and the top is given a spin $15\sqrt{\frac{3g}{b}}$, if the Top is slightly disturbed show that the axis will descend until it will make an angle $\cos^{-1}(7/11)$ with the vertical.



if slightly disturbed

$C^2\eta^2 > 4Mga A$ ⇒ S.H oscillation but if

$C^2\eta^2 < 4Mga A$ the top falls or descend.

(ii) Initial Conditions: at $t=0$, $\theta=0$, $\omega_3 = 15\sqrt{\frac{3g}{b}}$

$v_G = 0 = a\dot{\theta}\,\underline{i} + a\sin\theta\,\dot{\phi}\,\underline{j} = a\dot{\theta}\,\underline{i}$

⇒ $\dot{\theta} = 0$

(2) The geometry of the Top i.e moment of inertia G

$C = \frac{2}{5}(5m)b^2 + 0 = 2mb^2$

(mass of the sphere; sphere part; rod part)

$A = \frac{2}{5}(5m)b^2 + 5m(3b)^2 + \frac{1}{12}m(3b)^2$

(about I'; sphere part; rod part)

⇒ $A = 2mb^2 + 45mb^2 + 3mb^2 = 50mb^2$

$Ma = 5m \times 3b + m \times \frac{3b}{2} = \frac{33mb}{2}$

A. Mech.

3) Equations of Motion :

The three principal eqtns are:

$$\omega_3 = \eta \quad (1)$$

$$A\dot{\phi}\sin^2\theta + c\eta\cos\theta = H \quad (2)$$

$$A(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + 2Mga\cos\theta = K \quad (3)$$

From I. Conditions =>

$$\eta = 15\sqrt{\frac{39}{b}} \quad ; \quad H = c\eta \quad ; \quad K = 2Mga$$

From (2) in (3) =>

$$\dot{\phi} = [c\eta(1-\cos\theta)] / A\sin^2\theta$$

Substituting in eqtn (3) we find

$$A\dot{\theta}^2 + A\sin^2\theta \frac{c^2\eta^2(1-\cos\theta)^2}{(A\sin^2\theta)^2} + 2Mga\cos\theta = 2Mga$$

To find the limits of the motion we put $\dot{\theta} = 0$ =>

Therefore

$$(c^2\eta^2/A)(1-\cos\theta)^2 - 2Mga(1-\cos\theta)\sin^2\theta = 0$$

Since $\sin^2\theta = (1-\cos^2\theta) = (1-\cos\theta)(1+\cos\theta)$

$$\frac{c^2\eta^2}{A}(1-\cos\theta)^2 - 2Mga(1-\cos\theta)^2(1+\cos\theta) = 0 \Rightarrow$$

$$(1-\cos\theta)^2\left(\frac{c^2\eta^2}{A} - 2Mga(1+\cos\theta)\right) = 0$$

=> $\cos\theta = 1 \Rightarrow \theta = 0$ (double root = vertical position)

$$1+\cos\theta = \left(c^2\eta^2/2MgaA\right) =$$

$\Rightarrow 1 + \cos\theta = \left[4m^2b^2(1/5)^2 \; 3(9/5)\right] / 2g\left(\frac{35mb}{2}\right)(5mb^2)$

$= \frac{18}{11} \Rightarrow \cos\theta = \frac{15}{11} - \frac{11}{11} = \frac{7}{11}$

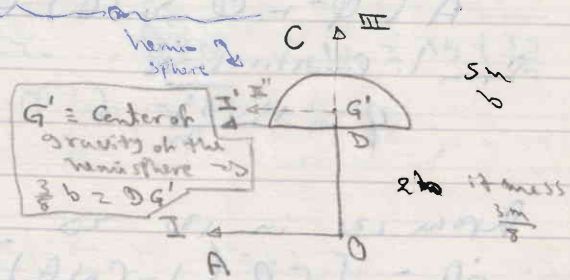
$\Rightarrow \theta = \cos^{-1}(7/11)$

* Note $c^2\eta^2/4mgaA = 9/11 \Rightarrow c^2\eta^2 < 4Mgah$ and a SHO oscillation about the vertical position is impossible.

$I_2)_{sphere} = \frac{2}{5}(2M)b^2 =$

$2I_2)_{hemisphere} \Rightarrow$

$I_2)_{hemisphere} = \frac{2}{5}Mb^2$



* M (mass of hemisphere)

$\Rightarrow C = \frac{2}{5}(5m)b^2 + 0 = 2mb^2$

hemisphere

$A = \frac{2}{5}(5m)b^2 - 5m(3b/8)^2 + 5m\left(\frac{3b}{8} + 2b\right)^2$

about I' (DG') about II''

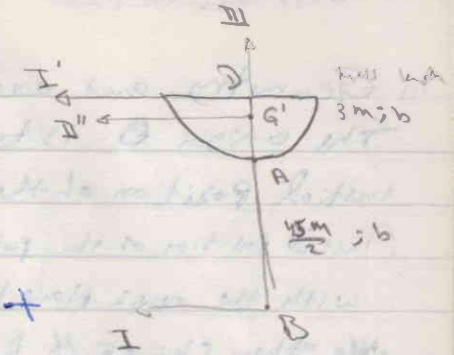
↳ moment of inertia about I for the hemisphere

$+ \frac{4}{3}\left(\frac{3m}{8}\right)(b^2) \Rightarrow$

↳ for the rod

$A = 2mb^2 + 5m\left(\frac{3b^2}{2} + 4b^2\right) + \frac{1}{2}mb^2 = 30\,mb^2$

$\therefore G' = \frac{3}{8}b$



$C = \frac{2}{5}(5m)b^2 = 2mb^2$

$\therefore G' = \frac{3}{8}b \quad ; \quad G'A = \frac{5}{8}b \Rightarrow$

$A = \frac{2}{5}(5m)b^2 - 5m\left(\frac{3}{8}b\right)^2 +$

$\Rightarrow$ $\Rightarrow$ about I' $\hookleftarrow$

$\Rightarrow$ about II'' $\hookleftarrow$

$5m\left(\frac{5}{2}b + b\right)^2 + \frac{4}{3}\,\frac{15}{2}m\left(\frac{b}{2}\right)^2 \Rightarrow$

for hemisphere $\hookleftarrow$ $\Rightarrow$ for the rod $\hookleftarrow$

$A = 2mb^2 + 5m\left(\frac{16}{64}b^2 + \frac{3}{4}b^2 + b^2\right) + \frac{15}{2}mb^2 =$

$22mb^2$.

Application (3)

The motion of a Sphere on a rough plane.

(Ex1) If the rough plane is fixed.

A uniform Solid sphere of radius a is Projected on a rough horizontal fixed plane of Coefficient of friction μ, if the Conditions at the biginning of the motion are such that Sliding (slipping) occurs initially (from) that the center of the sphere will describe a parabola for a time $t = 2V/7\mu g$ where V is the initial velocity of Sliding, show then that the rolling motion will start in the Center of gravity will move in a straight line with a uniform velocity.

Solid sphere

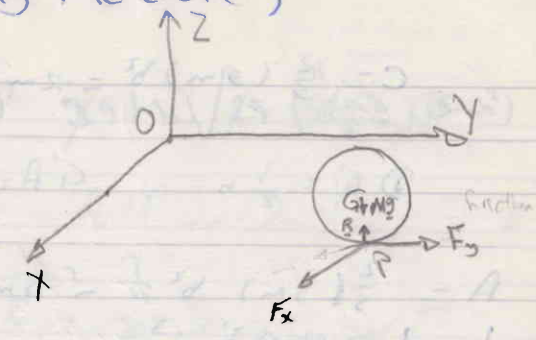
$I_x = \frac{2}{5}ma^2$

hollow Sphere

$I_x = \frac{2}{3}ma^2$

Plane fixed & axes are fixed with the plane

(Sliding Motion)



(1) Geometry and axes :

The origin O is taken at the initial position of the Sphere (initial position of the point of contact with the rough plane)

- We then choose a frame of reference (X, Y, Z) fixed in the space as follows : $(\hat{i}, \hat{j}, \hat{k})$

OX in the rough plane in the direction along direction initial of sliding

+ Since initial direction of the velocity of the point of contact P taking $OY \perp OX$ in the rough plane, $OZ \perp OXY$ of the plane.

(2) Position Velocity and acceleration of the Centre of gravity G (G is radius a)

$$\underline{r} = (x, y, a) = x\,\underline{i} + y\,\underline{j} + a\,\underline{k}$$

$$\underline{v}_G = \dot{\underline{r}} = (\dot{x}, \dot{y}, 0) \qquad \underline{f}_G = \ddot{\underline{r}} = (\ddot{x}, \ddot{y}, 0)$$

(at $t = 0$, $\dot{x} = u$; $t = 0$, $\dot{y} = v$)

(3) External Forces :

(1) (Mg) acting at G vertically down

(2) R Normal reaction acting at the Point of Contact (P) vertically upwards.

(3) The Friction $F = (F_x, F_y, 0)$ acting at P along the opposite direction of sliding (opposite direction of $\underline{v}_P$). Also from the Condition of sliding

$$F = \mu R = \sqrt{F_x^2 + F_y^2} \qquad (1)$$

(4) The Linear Motion:

$$M\ddot{\underline{r}} = \underline{F}_{external} \Rightarrow$$

$$M\ddot{x} = F_x \ (3); \quad M\ddot{y} = F_y \ (3); \quad M\ddot{z} = 0 = R - Mg$$

$$Mg = R \quad (4)$$

(5) Rotational Motion:

$$\dot{\underline{h}}_G = \underline{N}_G = \dot{\underline{h}}_G + \underline{q} \times M\ddot{\underline{r}} = \underline{N}_P$$

$$\ddot{\underline{r}} = \underline{f}_g \quad ; \quad \underline{q} \text{ Position vector of } G \text{ w.r. to } P$$

The rotational motion can be studied in one of the following 2 ways

(1) Rotational motion about G $\quad \dot{\underline{h}}_G = \underline{N}_G$

(2) the point of contact P

$$\dot{\underline{h}}_G + \underline{q} \times M\ddot{\underline{r}} = \underline{N}_P$$

The 1st method yields $\underline{h}_G = (A'\omega_1, B'\omega_2, C'\omega_3)$

A', B', C' moment of inertia about the axes x', y', z' passing through G parallel to x, y, z respectively. (Principal axes of inertia at G)

Hence $A' = B' = C' = \frac{2}{5} Ma^2$

for a solid sphere.

Therefore $\underline{h}_G = \frac{2}{5} Ma^2 \underline{\omega}$

$(\dot{x},\dot{y},0)$ (x,y,a) Friction

$$\Rightarrow \frac{2}{5} M a^2 \underline{\dot{\omega}} = \underline{N}_G = (\underline{r}_P - \underline{r}_G) \times \underline{F} =$$

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 0 & -a \\ F_x & F_y & 0 \end{vmatrix} = \frac{2}{5} a^2 \dot{\omega}_1 = a F_y \Rightarrow$$

$$F_y = \frac{2}{5} M a \dot{\omega}_1 \quad (5)$$

$$F_x = -\frac{2}{5} M a \dot{\omega}_2 \quad (6)$$

$\dot{\omega}_3 = 0$

(6) The Velocity of the Point of Contact.

$$\underline{v}_P = \underline{v}_G + \underline{v}_{PG} = \quad \text{i.e. } \underline{v}_P \text{ w.r.t. } P \text{ w.r.t. } G$$

$\underline{r}$

$$(\dot{x},\dot{y},0) + \underline{\omega} \times (\underline{r}_P - \underline{r}_G) = \quad \left[\begin{array}{l} \text{A st step if the} \\ \text{kind of motion is} \\ \text{not mentioned} \end{array}\right]$$

$(x,y,0)$ & (x,y,a)

$$(\dot{x},\dot{y},0) + \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \omega_1 & \omega_2 & \omega_3 \\ 0 & 0 & -a \end{vmatrix} = (\dot{x} - a\omega_2, \dot{y} + a\omega_1, 0) = \underline{v}_P \quad (8)$$

at $t=0$ $\dot{x} = u$, $\omega_2 = \Omega_2$

$\dot{y} = v$, $\omega_1 = \Omega_1$ $\Rightarrow$

[at $t=0$ $V = u - a\Omega_2$; $0 = v + a\Omega_1$] (9)

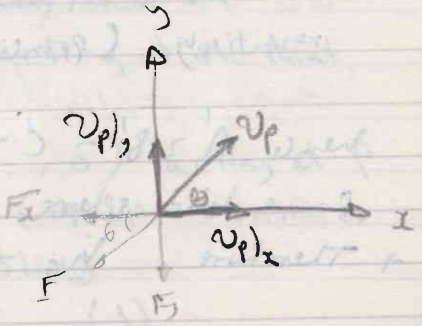
(the initial velocity of Sliding)

where the x-axis is taken along the initial direction of sliding $\Rightarrow$ and $v = 0$

Also from the Conditions of Sliding

from the Condition of Friction

$$\tan\theta = \frac{F_y}{F_x} = \frac{v_{P/y}}{v_{P/x}} = \frac{\dot{y} + a\omega_1}{\dot{x} - a\omega_2} \quad (10)$$



Also from eqt^{ns} (2), (3)

$$\frac{F_y}{F_x} = \frac{\ddot{y}}{\ddot{x}} \quad ; \quad \text{from (5), (6)} \quad \frac{F_y}{F_x} = \frac{a\dot{\omega}_1}{-a\dot{\omega}_2}$$

therefore $\frac{F_y}{F_x} = \frac{\ddot{y}}{\ddot{x}} = \frac{a\dot{\omega}_1}{-a\dot{\omega}_2}$ $\quad L.T^{-2}$ acceleration dimensions

$$\Rightarrow \frac{F_y}{F_x} = \frac{\ddot{y} + a\dot{\omega}_1}{\ddot{x} - a\dot{\omega}_2} \qquad (11) \quad \text{from (10) \& (11)}$$

$$\Rightarrow \frac{\ddot{y} + a\dot{\omega}_1}{\ddot{x} - a\dot{\omega}_2} = \frac{\ddot{y} + a\dot{\omega}_1}{\ddot{x} - a\dot{\omega}_2} \Rightarrow$$

$$\frac{\ddot{y} + a\dot{\omega}_1}{\dot{y} + a\omega_1} = \frac{\dot{x} - a\dot{\omega}_2}{\dot{x} - a\omega_2} \quad \text{integrating w.r.to time}$$

$$\ln(\dot{y} + a\omega_1) = \ln(\dot{x} - a\omega_2) + \text{Const.}$$

$$\Rightarrow \ln\left(\frac{\dot{y} + a\omega_1}{\dot{x} - a\omega_2}\right) = \text{Const.} \Rightarrow$$

$$\Rightarrow \frac{\dot{y} + a\omega_1}{\dot{x} - a\omega_2} = \text{Const.} = \frac{v_p)_y}{v_p)_x} = \tan\theta = \text{Const.} \qquad (12)$$

which is the direction of slipping

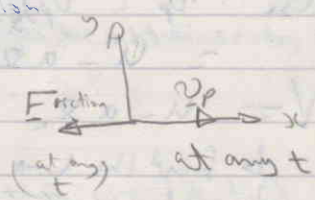
therefore the direction of slipping remains Const.

The Const in eqt^{n} (12) is determined from the initial conditions at $t = 0$ $\Rightarrow \tan\theta = (\dot{y} + a\omega_1)_{t=0} / (\dot{x} + a\omega_2)_{t=0} =$

$$= \frac{(V + a\Omega_1)}{V - a\Omega_2} = \frac{b}{V} = 0$$

i.e. sliding in the x-direction.

Therefore the sliding remains along the x-direction



we thus conclude that, $F_y = 0$ at any t

$$F_x = -\mu R$$

$$[F_x^2 + F_y^2]^{1/2} = R\mu \quad \text{at any time}$$

$\Rightarrow F_x = -\mu M g$. (14)

+ In order to find the time of the sliding motion we calculate v_p at any time and then take $v_p = 0$

We therefor substitute from eqt^{ns} (13), (14) in eqt^{ns} (2), (3), (5), (6)

From eqt^{n} (2) $M\ddot{x} = -\mu M g \Rightarrow \ddot{x} = -\mu g$

$\dot{x} = u - \mu g t$, $x = ut - \frac{1}{2}\mu g t^2$ (15)

also from eqt^{n} (3) $\ddot{y} = 0$, $\dot{y} = v$, $y = vt$ (16)

From eqt^{n} (5) $a\dot{\omega}_1 = 0 \Rightarrow a\omega_1 = a\Omega_1$ (17)

finally from eqt^{n} (6) $\frac{2M}{5} a\dot{\omega}_2 = -\mu g M$

$$a\dot{\omega}_2 = \frac{5}{2}\mu g \Rightarrow a\omega_2 = \frac{5}{2}\mu g t + \text{Const.} = \frac{5}{2}\mu g t + a\Omega_2 \quad (18)$$

* In order to find the eqt^{n} of motion during the sliding motion we substitute for t from eqt^{n} (16) in (15)

Therefore $x = X - ut = u\frac{y}{v} - \frac{1}{2}\mu g \frac{y^2}{v^2}$

$$-\frac{1}{2}\frac{\mu g}{v^2}\left(y^2 - \frac{2uv}{\mu g} y\right) = x$$

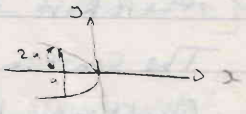
(which is similar to projectiles)

$$\left(y^2 - \frac{2uv}{\mu g} y\right) = -\left(\frac{2v^2}{\mu g}\right) x \quad (19)$$

by completing the square

$= 4a$

4a = semi-latus rectum



eqt^{n} (19) represent the eqt^{n} of Parabola of Semi

of Semi. Latus rectum equal to $= 2a = \frac{v^2}{\mu g}$

which proves that eqt^{n} of motion during

Sliding is a Parabola.

Also $v_p = v_p)_x = \dot{x} + a\omega_2 = u - \mu g t + a\Omega_2 =$

$= u - a\Omega_2 - \frac{7}{2}\mu g t$

$V = u - a\Omega_2$ = initial velocity of sliding

to find the time of sliding $v_p = 0$ ⇒

$$\boxed{t = \frac{2V}{7\mu g}}$$ the required time of sliding (Ans)

A. Mech.

Note for pure rolling motion $\mu \geq \frac{K_G^2}{K_G^2 + a^2} \tan\alpha$

K_G radius of gyration, α = inclination of the plane to the horizontal

Note: The rotational motion may be studied about the point P without

in this case we have to use eqtn

$$\underline{\dot{h}}_G + \underline{q} \times M\underline{\ddot{r}} = \underline{M}_P = (\underline{r} - \underline{r}_P) \times (-N_y \underline{\kappa})$$

$\underline{q}$ position vector G, w.r.t P. of F_A

$$\underline{\kappa} \times \underline{\kappa} \quad M_y \underline{\kappa} = 0$$

Therefore $\underline{\dot{h}}_G + \frac{2}{5} M a^2 \underline{\omega} + \begin{vmatrix} \underline{i} & \underline{j} & \underline{\kappa} \\ 0 & 0 & a \\ \dot{x} & \dot{y} & 0 \end{vmatrix} = 0 \Rightarrow$

$\frac{2}{5} a \dot{\omega}_z - \ddot{y} = 0$ eqtn (3), (5) (5)'

$\frac{2}{5} a \dot{\omega}_z + a\ddot{x} = 0$ eqtn (4), (6) (6)'

+ The pure rolling Motion:

If friction is sufficient to prevent sliding then the pure rolling motion occurs when $v_p = 0$, the required conditions

$$\mu > \left(K_G^2 / (K_G^2 + a^2)\right) \tan\alpha$$

K_G = radius of gyration ; α = the inclination of the plane to the horizontal.

In the case of a horizontal plane $\alpha = 0$ and the condition is always satisfied since $\mu > 0$

The initial conditions of the pure rolling motion can be obtained by substituting in eqtns (15) to (18) for

$$t = \frac{I}{2} \frac{V}{\mu g}$$

As regards the equations of motion they are identical with eqtns (2) to (7)

eqtn (1) must be excluded

1. The Kinematical Conditions:

$\underline{v}_p = 0 \xrightarrow{\text{gives}} \dot{x} - a\omega_2 = 0 \Rightarrow \dot{x} = a\omega_2$ (21)

$\dot{y} + a\omega_1 = 0 \quad \dot{y} = -a\omega_1$ (22)

Also from eqtn (2), (6) or (6)' we have

$\ddot{x} = -\frac{2}{5} a\dot{\omega}_2 = -\frac{2}{5}\ddot{x}$

« from Kinematical Conditions (21) »

$\frac{7}{5}\ddot{x} = 0$ i.e $\ddot{x} = 0 \Rightarrow \dot{x} = \text{Const.} = v_1$ determined from eqtn (15).

$x = v_1 t + x_1$ x_1 determined from eqtn (1)

Similarly from (3), (5) or (5)' we have

$\ddot{y} = \frac{5}{2} a\dot{\omega}_1 = -\frac{2}{5}\ddot{y} \Rightarrow \frac{7}{5}\ddot{y} = 0 \Rightarrow \ddot{y} = 0 \Rightarrow \dot{y} = \text{Const.} = v_1$

From K. Conditions (22) ; $y = v_1 t + y_1$

v_1 , y_1 from eqtn (16) (24)

from (23), (24) $\frac{y - y_1}{x - x_1} = \frac{v_1 t}{v_1 t} = \frac{v_1}{v_1}$ (25)

which is the eqtn of a straight line with a slope $\frac{v_1}{v_1}$ with the x axis.

Note: The rotational motion can be studied about the point of contact p the resulting eqtns are (5)', (6)'

Moreover, in this case (pure rolling) $\underline{v}_p = 0 \Rightarrow$ p can be considered fixed point instantaneously

i.e p is the instantaneous center of rotation (instantaneously fixed)

We may thus use the eqtn $\underline{\dot{h}}_p = \underline{M}_p$ (pure rolling) and fixed plane

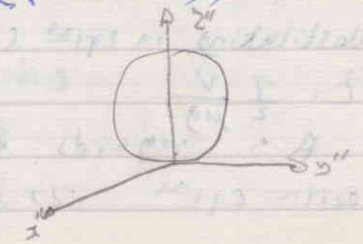
$I_{z''} = \frac{2}{5} Ma^2$, $I_{y''} = \frac{7}{5} Ma^2$

$I_{x''} = \frac{7}{5} Ma^2 \omega_1$

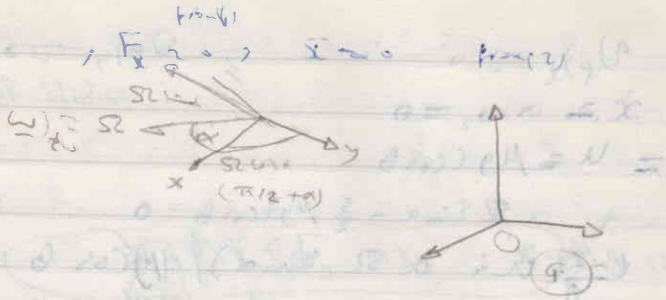
$\left(\frac{7}{5} Ma^2 \dot{\omega}_1 , \frac{7}{5} Ma^2 \dot{\omega}_2 , \frac{2}{5} Ma^2 \dot{\omega}_3\right) = 0$

$a\dot{\omega}_1 = 0$, $a\dot{\omega}_2 = 0$, $\dot{\omega}_3 = 0$

(5)'' (6)''



From (5) $F_y = 0$, $\ddot{y} = 0$; From (1) $F_x = 0$, $\ddot{x} = 0$ From (2)



2. Position, velocity and acceleration of G.

$\dot{\underline{r}} = (\dot{x}, \dot{y}, 0)$; $\ddot{\underline{r}} = (\ddot{x}, \ddot{y}, 0)$

* Velocity of the point of contact

$\underline{v}_p = \underline{v}_G + \overset{\omega \times (r_p - r)}{\underline{\omega} \times \underline{r}_{PG}} = (\dot{x} - a\omega_2, \dot{y} + a\omega_1, 0) =$

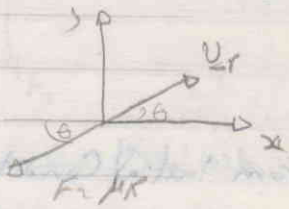
$u + a\Omega \sin\alpha \neq 0$ $\quad v_p)_{y,t=0} = a\Omega\cos\alpha \neq 0$

$v_p|_{t=0} \neq 0$ and the motion starts with sliding

$\underline{v}_p)_{t=t_0} = \frac{F_y}{F_x} = \frac{v_p)_y}{v_p)_x}$

3. The sliding Motion.

$$\tan\theta = \frac{\dot{y} + a\omega_1}{\dot{x} - a\omega_2} = \text{Const.} = \frac{v_p)_{y,t=0}}{v_p)_{x,t=0}} = \frac{a\Omega\cos\alpha}{u + a\Omega\sin\alpha}$$



$F_x = -\mu R\cos\theta = -\mu\mu_g\cos\theta$

$F_y = -\mu\mu_g\sin\theta$

Substituting in eqt^{n} (2), (3), (5), (6)

given

$\ddot{x} = F_x = -\mu R\cos\theta$;

$\dot{x} = u - \mu g t\cos\theta$, $x = ut - \frac{1}{2}\mu g t^2\cos\theta$

$\ddot{y} = -\mu g\sin\theta$, $\dot{y} = \text{Const.} = -\mu g t\sin\theta$

$y = -\mu g\frac{t^2}{2}\sin\theta + \text{Const.}$

$a\dot{\omega}_2 = \frac{5}{2}\mu g - \Omega$ [illegible] $\Rightarrow a\omega_2 = -a\Omega\sin\alpha + \frac{5}{2}t\mu g$

$a\omega_1 = -a\Omega\cos\alpha - \frac{5}{2}t\mu g$.

To find the time of sliding let $\underline{v}_p = 0$

$(v_p)_x = 0$; $(v_p)_y = 0$

$\dot{x} = a\omega_2 = 0$

$= u - \mu g t \cos\theta$

$+ a\Omega \sin\alpha - \frac{5}{2}\mu g t \cos\theta = 0$

$t = \frac{2}{7}(u + a\Omega \sin\alpha)/\mu g \cos\theta$ $\quad a\Omega\cos\alpha / \mu\cos\theta$

$(v_p)_y = 0 \Rightarrow \dot{y} + a\omega_1 = 0$

$-\mu g t \sin\theta + a\Omega\cos\alpha - \frac{5}{2}\mu g t \sin\theta = 0$

$\dot{x} + a_2\omega = 0$

$u - \mu g t \cos\theta + a\Omega\sin\alpha - \frac{5}{2} t \mu g \cos\theta$

From Initial Conditions:

$\tan\theta = a\Omega\cos\alpha / u + a\Omega\sin\alpha = \frac{(v_p)_y}{(v_p)_x}\Big|_{t=0} = \frac{V\sin\theta}{V\cos\theta}$

$\frac{a\Omega\cos\alpha}{\sin\theta} = \frac{u + a\Omega\sin\alpha}{\cos\theta} = V$ $\Rightarrow$

direction of v_p at $t=0$

A. Mech.

The eqtns of motion of sliding:

$F_x^2 + F_y^2 = \mu^2 R^2$ (1)

$M\ddot{x} = F_x$ (2) ; $M\ddot{y} = F_y$ (3) ; $R = Mg$ (4)

$\underline{N}_G = \underline{N}_G \Rightarrow \underline{N}_G + \underline{q} + M\underline{F} = \underline{N}_P$

$\underline{q} \equiv G$ w.r.t. O

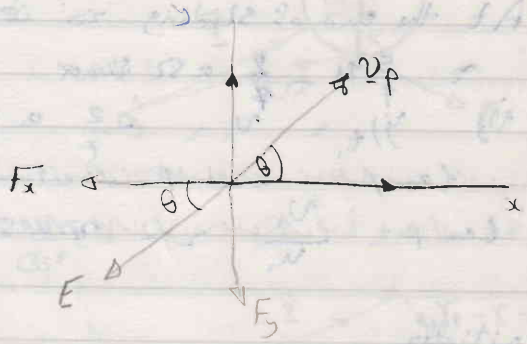
$\frac{2}{5} Ma\ddot{w}_1 = +F_y$ (5) ; $\frac{2}{5} Ma\ddot{w}_2 = -F_x$ (6) ; $\dot{w}_3 = 0$

From (5) $\Rightarrow \frac{\dot{y} + a\dot{w}_1}{\dot{x} - a_2 w_2} = \text{Const.}$;

$F_y / F_x = (v_P)_y / (v_P)_x = \text{Const.} = (v_P)_y / (v_P)_x \Big|_{t=0} = \tan\theta$

Therefore, at any t

$F_x = -\mu R \cos\theta = -\mu M$



Subst. in the eqtns of Motion

2, 3, 5, 6)

$\ddot{x} = -\mu g \cos\theta$ (7)

$\dot{x} = -(\mu g \cos\theta)\, t + u$ (7) ; $x = ut - \frac{1}{2}\mu g t^2 \cos\theta$

$\ddot{y} = -\mu g \sin\theta$; $\dot{y} = -\mu g t \sin\theta$ (8)

$\Rightarrow a\dot{w}_1 = +\frac{5}{2}F_y\left(-\frac{5}{2}\mu g \sin\theta\right) \Rightarrow$

$a w_1 = -\frac{5}{2}\mu g t \sin\theta + a\Omega_2 \cos\alpha$. (9)

$a\dot{w}_2 = -\frac{5}{2}(-\mu g \cos\theta) = \frac{5}{2}\mu g \cos\theta$. $\Rightarrow$ initially

$a w_2 = -a\Omega_2 \sin\alpha + \frac{5}{2}\mu g t \cos\theta$ (10)

In order to find the time of sliding we must put $\underline{v}_P = 0$

i.e. $(v_P)_x = 0$ & $(v_P)_y = 0$

i.e. $\dot{x} - a w_2 = 0 \Rightarrow$

Putting $(v_P)_x = 0$ implies $u = \mu g t \cos\theta + a\Omega_2 \sin\alpha - \frac{5}{2}\mu g t \cos\theta$

$\Rightarrow t = \frac{1}{2}(u + a\Omega_2 \sin\alpha) / \frac{1}{2}\mu g \cos\theta$ = (time of sliding)

To get the required solution, then using $\Rightarrow$ as

$$t = \frac{2V}{7\mu g} \cdot \qquad V \text{ (initial velocity of sliding)}$$

$$t = \frac{2V}{7\mu g} = \frac{2a\Omega\cos\alpha}{7\mu g \sin\beta} \cdot$$

(y) $v_{P|y} = \dot{y} + a\omega_1 = -\mu g t \sin\beta + a\Omega\cos\alpha - \frac{5}{2}\mu g t \sin\beta$.

giving the above found answer for t.

* The 3rd required point $\Rightarrow$

At the end of sliding $\Rightarrow$ $\dot{x}|_{t_1} = u_1 = u - \frac{2}{7}(u + a\Omega\sin\alpha)$

$= \frac{5}{7}u - \frac{2}{7}a\Omega\sin\alpha$

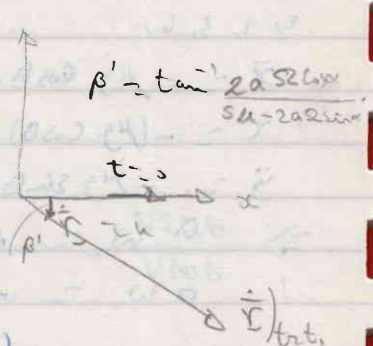
(y) $\dot{y}|_{t_1} = v_1 = -\frac{2}{7}a\Omega\cos\alpha$.

$\tan\beta$ and β direction between of $\dot{r}$ at $t = t_1$

$\tan\beta = \frac{v_1}{u_1} = -2a\Omega\cos\alpha / 5u - 2a\Omega\sin\alpha$.

Initially

$\beta' = \tan^{-1}\frac{2a\Omega\cos\alpha}{5u - 2a\Omega\sin\alpha}$.



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** In the 1st example (Pure rolling motion part)

$\ddot{x} = 0$ (23), ; $\ddot{y} = 0$ (24)

$\Rightarrow$ (23), (24) $\Rightarrow$ $F_x = 0$ $\Rightarrow$ $\mu \geq \frac{k^2}{k^2 + a^2}\tan\alpha$

$F_x = M\ddot{x} = 0$; $F_y = M\ddot{y} = 0$ $\Rightarrow$ $F = 0$

$\therefore \frac{F}{R} = 0 < \mu$ (hence friction is not limited)

this confirms that the motion is indeed a pure rolling one.

* in the note after pure rolling motion: (5'), (6'):

$\dot{h}_G + \underline{q} \times M\ddot{\underline{r}} - \underline{N}_p = 0$ ... → (5)'', (6)''

$a\dot{\omega}_1 = 0$ (5)'' ; $aq\dot{\omega}_2 = 0$ (6)''

From kinematical conditions: $\dot{X} = a\omega_2$ (2.1) ; $\dot{Y} = a\omega_1$ (2.2)

$\dot{Y} = 0$ from (5)'' → $\omega_1 = 0$

also (2.1) gives $\dot{x} = 0$ (2.3)

Therefore, eqt^ns (5), (6) ≡ (5)', (6)' ≡ (5)'', (6)''.

recalling from $\dot{h}_G = \underline{N}_G$ (Pure rolling) $\dot{h}_p = \underline{N}_p = 0$

+ An alternative solution for example Grid:-

hence eqt^ns of motion

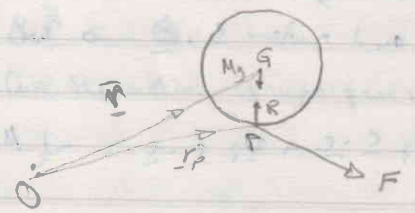
$R = Mg$ (1) ; $M\ddot{\underline{r}} = \underline{F}$ (2)

which is Linear motion.

(Also Rotational Motion about G)

is $\dot{h}_G = \underline{N}_G$.

$I\dot{\underline{\omega}} = (\underline{r}_p - \underline{r}) \times \underline{F} = -a\underline{K} \times \underline{F}$ (3)'

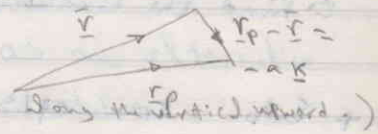


O is the origin in the rough plane which is the initial position of p.

$I = \frac{2}{5}Ma^2 = A = B = C$

$\Rightarrow$

$I\dot{\underline{\omega}} + a\underline{K} \times M\ddot{\underline{r}} = 0$;



($\underline{K}$ = unit vertical (static) upward)

$\dot{h}_a$ $\underline{q} = \underline{r} - \underline{r}_p$ (4) ≡ rotational motion about p.

$\frac{2}{5}a\dot{\underline{\omega}} + \underline{K} \times \ddot{\underline{r}} = 0$ (4)' ∴ H

The initial conditions causes a sliding motion at 1,1'

+ The Sliding Motion:

$\underline{F} = -\mu R\, \underline{e}$ (5) ; $\underline{e}$ unit vector along $\underline{v}_p$ at anytime t.

hence $\underline{e}$ is a rotating unit vector.

$\Rightarrow \underline{F} = -\mu Mg\, \underline{e}$. eqt^ns (5), (2)

$\ddot{\underline{r}} = -\mu g\, \underline{e}$ (6)

Also the velocity of the point of contact $\underline{v}_p = \underline{v}_G + \underline{v}_{pG}$ [illegible]

$v_p = \underline{v}_p \underline{e} = \dot{\underline{r}} + \underline{\omega} \times (\underline{r}_p - \underline{r}) = \dot{\underline{r}} + \underline{\omega} \times (-a\underline{K})$.

hence $\underline{v}_p = v_p \underline{e} = \dot{\underline{r}} + \underline{\Omega} \times \underline{w}$ (7)

differentiating eqt^n (7) w.r.t. to t, we get

$\dot{\underline{v}}_p = \dot{v}_p \underline{e} + v_p \dot{\underline{e}} = \ddot{\underline{r}} + \underline{\alpha} \times \underline{w}$ (8)

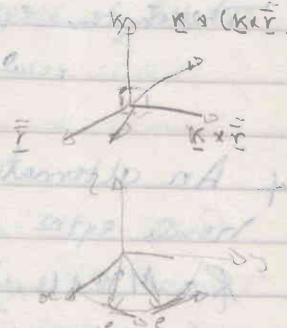
From (4) in (8) for $\underline{w}$ we find

$\ddot{\underline{r}} + \underline{\alpha} \times \left(-\frac{5}{2} \underline{K} \times \ddot{\underline{r}}\right) = \ddot{\underline{r}}^{*} - \frac{5}{2}\{(\underline{K} \cdot \ddot{\underline{r}})\underline{K} - (\underline{K} \cdot \underline{K})\ddot{\underline{r}}\}$

$\ddot{\underline{r}}$ always in the rough plane $\Rightarrow \underline{K} \perp$ rough plane $\Rightarrow$

$= \frac{7}{2} \ddot{\underline{r}}$ $\quad \because \|\underline{K} \times \ddot{\underline{r}}\| \perp \underline{K}, \ddot{\underline{r}}$ $\quad = \|\underline{i}\|$

From (6) $\Rightarrow$ $-\frac{7}{2} \mu g \underline{e} = \frac{7}{2} \ddot{\underline{r}}$ (9)



Taking the scalar product of eqt^n (9) with $(\underline{e})$ and note $\underline{e} \cdot \dot{\underline{e}} = 0$ we find

$\dot{v}_p \underline{e} \cdot \underline{e} + v_p \dot{\underline{e}} \cdot \underline{e} = -\frac{7}{2} \mu g \underline{e} \cdot \underline{e}$ $\quad \underline{\Omega}$ angular velocity of $\underline{e}$

$\dot{\underline{e}} = \underline{\Omega} \times \underline{e}$

$\Rightarrow \underline{\Omega} \perp \underline{e}$ & $\perp \dot{\underline{e}}$

$\dot{v}_p = -\frac{7}{2} \mu g$ (10) ; V initial velocity of sliding.

$v_p = V - \frac{7}{2} \mu g t$ (11)

to find the time of sliding put $v_p = 0 \Rightarrow \boxed{t = t_1 = \frac{2V}{7\mu g}}$

+ Subs. from eqt^n (10) in eqt^n (9) $\Rightarrow$

$\dot{v}_p \underline{e} + v_p \dot{\underline{e}} = -\frac{7}{2} \mu g \underline{e} \Rightarrow \dot{\underline{e}} = 0$; $\underline{e}$ = const. vector along the initial direction.

+ From eqt^n (6) $\ddot{\underline{r}} = -\mu g \underline{e}$; $\underline{e}$ const. vector.

$\dot{\underline{r}} = \dot{\underline{r}}_0 - \mu g t \underline{e}$ $\Rightarrow$ if $t = t_1 \Rightarrow \dot{\underline{r}} = \dot{\underline{r}}_1$ (giving) (Using initial conditions etc.)

$\underline{r} = \underline{r}_0 t - \frac{1}{2} \mu g t^2 \underline{e}$ $\Rightarrow$ leads to an eqt^n of a parabola.

+ Rolling Motion:

* Rolling Motion: eqtn (5) & (6) must

$\underline{v}_p = \dot{\underline{r}} + a\underline{K} \times \underline{\omega}$ (7) from eqtn (7) $\Rightarrow \dot{v}_p = 0$

which is Condition of Rolling Motion.

Differentiating w.r.t. t. $\Rightarrow$

$\ddot{\underline{r}} + a\underline{K} \times \dot{\underline{\omega}} = 0$ subs. from (4) $\Rightarrow$

$\ddot{\underline{r}} + a\underline{K} \times \left(-\frac{5}{2} \underline{K} \times \ddot{\underline{r}} \right) = 0$

$+\frac{5}{2}\ddot{\underline{r}}$

$\Rightarrow \frac{7}{2}\ddot{\underline{r}} = 0 \Rightarrow \ddot{\underline{r}} = 0$; $\Rightarrow \dot{\underline{r}} = \text{const.} = \dot{\underline{r}}_1$

for (initial condition of rolling) $\Rightarrow \{ \dot{\underline{r}} = \dot{\underline{r}}_1, t = \underline{r}_1 \}$ (12)

which is an eqtn of a straight line.

$\Rightarrow$ eqtn (12), (2) gives $F = M\ddot{\underline{r}} = 0 \Rightarrow$

$F/R = 0 < \mu$ therefore the motion is indeed pure rolling.

Analitical Dynamics

Sofar we have dealt with the problems in Dynamics by using Newton's Laws of motion, as an alternative approach Analytical Dynamics may be utilized. This new approach (1760) has the advantage of that its results can also be applied to other branches of physics (Quantum Mechanics, Statistical Mechanics, Solid state Phy, ...)

Generalized Coordi.

The generalized Coordinates of a dynamical system are the min. number of independent parameters needed to specify the motion of the system, they are usually denoted by $(q_1, q_2, \ldots, q_n)$, where n is the Number of generalized Coordinates

not necessarily have the dimension of length.

They don't have required the dimension of length

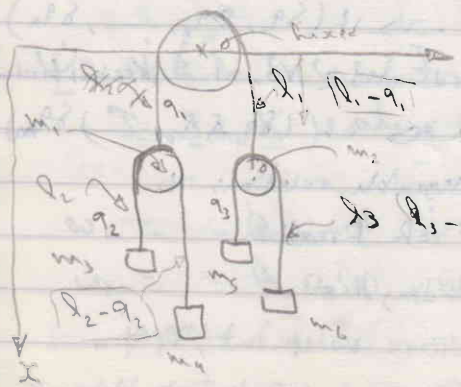
example 1. For a particle moving in a plane we need two generalize Coordinates which may be taken as

$(x, y) \equiv (q_1, q_2)$; $(r, \theta) = (q_1, q_2)$ or $(\lambda, \psi) = (q_1, q_2)$

Cartisian polar intrinsic

example 2. For the plane motion of a rigid body we normally use $(\bar{x}, \bar{y})$ (q_1, q_2) three generalized Coordinates mainly namely x to describe the linear motion of the Center of gravity G and $\theta = q_3$ to determine the rotational motion about G

example 3. The System of Pulleys shown in fig. is determined entirely by the three generalized Coordinates (q_1, q_2, q_3)



$\Rightarrow x_1 = q_1$, $x_2 = l_2 - q_1$

$x_3 = q_1 + q_2$, $x_4 = q_1 + l_2 - q_2$

$x_5 = l_1 - q_1 - q_3$

$x_6 = l_1 - q_3 + l_3 - q_3$.

Generalized Velocities and accelerations:

The quantities $(q_1, q_2, \ldots, q_n)$, $(\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n)$ are turned, respectively, to the generalized velocities and accelerations. They don't in general have the dimensions of velocities and accelerations.

Holonomic and non Holonomic Systems.

Let $(q_1, q_2, \ldots, q_n)$ be the generalized Coordinates of the system and let $(\delta q_1, \delta q_2, \ldots, \delta q_n)$ be the infinitesimal changes in these coordinates in δt.

If $(\delta q_1, \delta q_2, \ldots, \delta q_n)$ are independent then the system is said to be Holonomic, on the other hand if the infinitesimal changes $(\delta q_1, \delta q_2, \ldots, \delta q_n)$ are related by some non-integrable relations (say, m relations) which result from some Constraints then the system is non-Holonomic.

i.e. $\alpha_1(q_1, q_2)\delta q_1 + \alpha_2(q_1, q_2)\delta q_2 = 0$

$\Rightarrow$ if we can integrate $\Rightarrow$ q_1 can be related to q_2, etc.

The number $n - m$ is usually called the number of degrees of freedom

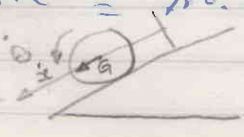
No. of Gen. Coor. — No. of relations.

The Holonomic system is thus characterized by the fact that the No. of generalized Coordinates is equal to the No. of degrees of freedom.

Example: $(q_1, q_2, \ldots, q_n)$ independent. $\Rightarrow$ if $(\delta q_1, \delta q_2, \ldots, \delta q_n)$ $\equiv$ Holonomic system. No of G. Coordinates $\equiv$ No. of degrees of freedom.

$(q_1, q_2, \ldots, q_n)$ independent $\Rightarrow$ if $(\delta q_1, \delta q_2, \ldots, \delta q_m)$ s.t they are related by m nonintegrable relations $\Rightarrow$

$n - m \equiv$ No. of degrees of freedom.



Rotational motion

Sliding motion

$\dot{x}, \dot{\theta}$ are independent, $q_p \neq 0$.

$\delta x, \delta\theta$ are independent $\Rightarrow$ Holonomic.

Pure rolling $v_p = 0 \Rightarrow \dot{x} = a\dot{\theta} \Rightarrow \delta x = a\,\delta\theta$ (integrable relation)

$\Rightarrow$ integrating $x = a\theta + C \Rightarrow q_1, q_2$ are not independent.

Holonomic System of two G. Coordinates $\Rightarrow$ the system is holonomic in one generalized coordinate.

Example: Consider the motion of a sphere on a horizontally rough plane. In order to describe the motion five generalized coordinates are needed (x, y) to describe the linear motion of the center of gravity G. and the three Eulerian angles (θ, ϕ, ψ) to determine the angular velocity $\underline{\omega}$ (rotational motion about G). We can the distinguish between the following two cases.

(i) Sliding Motion $(v_p \neq 0)$ friction is not enough to prevent sliding the infinitesimal changes $(\delta x, \delta y, \delta\theta, \delta\phi, \delta\psi)$ are independent and the system is holonomic. The No of degrees of freedom equals five.

(ii) Pure rolling motion. $\underline{v}_p = 0$ (friction is sufficient to prevent sliding) which is a constraint. The infinitesimal changes $(\delta x, \delta y, \delta\theta, \delta\phi, \delta\psi)$ are no longer independent since they must be related due to the constraints $\underline{v}_p = 0$ which gives

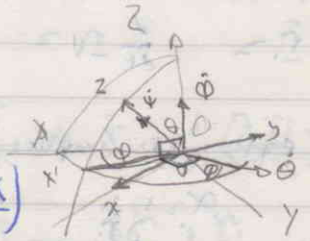
$\underline{v}_p = \underline{v}_G + \underline{v}_{pG} = 0 \Rightarrow$

$\dot{x} - a\omega_2 = 0 \quad , \quad \dot{y} + a\omega_1 = 0 \qquad (1)$

$\dot{\underline{\phi}} = \dot{\phi}\underline{K} \; ; \; \dot{\underline{\theta}} = -\dot{\theta}\sin\phi\,\underline{I} + \dot{\theta}\cos\phi\,\underline{J}$

$\dot{\underline{\psi}} = \dot{\psi}(\sin\theta\cos\phi\,\underline{I} + \sin\theta\sin\phi\,\underline{J} + \cos\theta\,\underline{K})$

$(\underline{I}, \underline{J}, \underline{K})$ along (X, Y, Z)



$\omega_1 = -\dot{\theta}\sin\phi + \dot{\psi}\sin\theta\cos\phi$

$\omega_2 = \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi \qquad (2)$

eqtns (1), (2) give

$\dot{x} - a\dot{\theta}\cos\phi - a\dot{\phi}\sin\theta\sin\phi = 0$

$\dot{y} - a\dot{\theta}\sin\phi + a\dot{\phi}\sin\theta\cos\phi = 0$

$$\left.\begin{aligned} \delta x - a\,\delta\theta\cos\phi - a\,\delta\phi\sin\theta\sin\phi &= 0 \\ \delta y - a\,\delta\theta\sin\phi + a\,\delta\phi\sin\theta\cos\phi &= 0 \end{aligned}\right\} \quad (4)$$

are two non-integrable relations between $(\delta x, \delta y, \delta\theta, \delta\phi, \delta\psi)$ $(m = 2)$

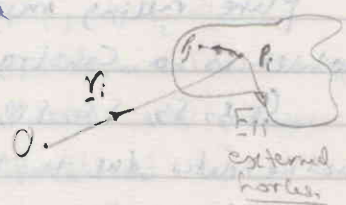
the system is non-holonomic, and the no. of degrees of freedom $= 5 - 2 = 3$

Lagranges eqtns for a Holonomic System.

We Consider a holonomic dynamical system of particles $(P_1, P_2, \ldots)$ of masses $(m_1, m_2, \ldots)$ let $(q_1, q_2, \ldots, q_n)$ be the generalized coordinates of the system. (The No. of g. coord. = No. of degrees of freedom = n)

The linear eqtn of motion of any particle

$$m_i \ddot{\underline{r}}_i = \sum_j \underline{F}_{ij} = \underline{F}_i \quad (1)$$



Since $(q_1, q_2, \ldots, q_n)$ are the generalized coordinates of the system then $\underline{r}_i = \underline{r}_i(q_1, q_2, \ldots, q_n, t)$ (2)

eqtn (2) determines the trajectory of the particle P_i, from this eqtn

$$\dot{\underline{r}}_i = \frac{d}{dt}\underline{r}_i = \sum_{\alpha=1}^{n} \left(\partial \underline{r}_i / \partial q_\alpha\right) \cdot \dot{q}_\alpha + \frac{\partial \underline{r}_i}{\partial t} \quad (3)$$

(i.e. a f^n of q_α, t)

eqtn (3) implies that $\dot{\underline{r}}_i$ is a f^n of $q_\alpha, \dot{q}_\alpha, t$ $\alpha = 1, 2, \ldots n$ (as well as $\dot{q}_\alpha$, $\alpha = 1, 2, \ldots n$)

$$\therefore \frac{\partial \dot{\underline{r}}_i}{\partial \dot{q}_s} = \text{since } \underline{r}_i = \underline{r}_i(q, t) \Rightarrow$$

$$\frac{\partial \dot{\underline{r}}_i}{\partial \dot{q}_s} = \frac{\partial \underline{r}_i}{\partial q_s} \quad (4)$$

the Coefficient of $\dot{q}_s$ in eqtn (3) $= \left(\partial \underline{r}_i / \partial q_s\right)$

$\frac{\partial^2}{\partial x \partial y} = \frac{\partial^2}{\partial y \partial x}$

The next step in the derivation is to show that

$$\frac{d}{dt}\left(\frac{\partial \underline{r}_i}{\partial q_s}\right) = \frac{\partial}{\partial q_s}\dot{\underline{r}}_i = \frac{\partial}{\partial q_s}(d\underline{r}_i/dt) \quad (5)$$ i.e. the Commute

(the two operators $\frac{d}{dt}$, $\frac{\partial}{\partial q_s}$ are Commutative.)

To prove this we note that $\frac{\partial \underline{r}_i}{\partial q_s}$ is a fn of $\underline{q}, t$.

Therefore $\frac{d}{dt}\left(\frac{\partial \underline{r}_i}{\partial q_s}\right) = \sum$

$$\sum_{\alpha=1}^{n} \frac{\partial^2 \underline{r}_i}{\partial q_\alpha \partial q_s}\dot{q}_s + \frac{\partial^2 \underline{r}_i}{\partial t \partial q_s} = \left[\sum_{\alpha=1}^{n} \frac{\partial^2 \underline{r}_i}{\partial q_s \partial q_\alpha} + \frac{\partial^2 \underline{r}_i}{\partial q_i \partial t}\right] =$$

$$= \frac{\partial \dot{\underline{r}}_i}{\partial q_s} \quad (5)$$

The kinetic energy of the system $T = \frac{1}{2}\sum_{i=1} m_i \dot{\underline{r}}_i \cdot \dot{\underline{r}}_i$ (6)

$\underline{r}_i = \underline{r}_i(\underline{q}, t) \Rightarrow T(\underline{q}, t) = T$.

Therefore

$$\frac{\partial T}{\partial q_s} = \frac{1}{2} \times 2 \sum_i m_i \dot{\underline{r}}_i \cdot \frac{\partial \dot{\underline{r}}_i}{\partial q_s} = \sum_{i=1} m_i \dot{\underline{r}}_i \cdot \frac{\partial \dot{\underline{r}}_i}{\partial q_s} \quad (7)$$

Also $\frac{\partial T}{\partial \dot{q}_s} = \sum m_i \dot{\underline{r}}_i \cdot \frac{\partial \underline{r}_i}{\partial q_s}$ but from (4) $\Rightarrow$

$$= \sum m_i \dot{\underline{r}}_i \cdot \frac{\partial \underline{r}_i}{\partial q_s} \quad (8)$$

differentiating w.r to time $\Rightarrow$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) = \sum_i m_i \ddot{\underline{r}}_i \cdot \frac{\partial \underline{r}_i}{\partial q_s} + \sum_{i=1} m_i \dot{\underline{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \underline{r}_i}{\partial q_s}\right) =$$

$$\text{from (5)} \quad + \sum_{i=1} m_i \dot{\underline{r}}_i \cdot \frac{\partial \dot{\underline{r}}_i}{\partial q_s}$$

from (1, 7) $\Rightarrow$

$$= \sum \underline{F}_i \cdot \frac{\partial \underline{r}_i}{\partial q_s} + \frac{\partial T}{\partial q_s} \quad \text{(from 7)}$$

from (1)

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_s} - \frac{\partial T}{\partial q_s} = \sum \underline{F}_i \cdot \frac{\partial \underline{r}_i}{\partial q_s} = \text{Generalized Components of Force.}$$

$\frac{\partial T}{\partial \dot{q}_s}$ Generalized Components of momentum

A. Mech.

$\underline{\Omega} = (-\dot\varphi \sin\theta, \dot\theta, \dot\varphi \cos\theta)$

angular velocity of the axes $= \underline{\omega}$ (angular velocity of the rod)

$T = \frac{1}{2} \underline{h}_o \cdot \underline{\omega}$, kinetic energy

$\underline{h}_o = (A\omega_1, B\omega_2, C\omega_3)$, $C = 0$, $A = B = \frac{4}{3} M a^2$

$\Rightarrow T = \frac{2}{3} M a^2 (\dot\varphi^2 \sin^2\theta + \dot\theta^2)$

potential energy $V = -Mg\, a \cos\theta$

Lagrangian $= L = T - V = \frac{2}{3} M a^2 (\dot\theta^2 + \dot\varphi \sin^2\theta) + Mg a \cos\theta$ (3)

which is the Lagrangian function.

In order to apply Lagrange's equations we have to calculate

$\frac{\partial L}{\partial \dot\theta}$, $\frac{\partial L}{\partial \dot\varphi}$, $\frac{\partial L}{\partial \varphi}$, $\frac{\partial L}{\partial \theta}$

$\frac{\partial L}{\partial \dot\theta} = \frac{4}{3} M a^2 \dot\theta$; $\frac{\partial L}{\partial \theta} = \frac{4}{3} M a^2 \dot\varphi^2 \sin\theta\cos\theta - Mg a \sin\theta$ (4)

$\frac{\partial L}{\partial \dot\varphi} = \frac{4}{3} M a^2 \dot\varphi \sin^2\theta$; $\frac{\partial L}{\partial \varphi} = 0$

We then substitute from eq^{n} (4) in the Lagrangian eq^{n}

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = 0$

$\therefore \frac{4}{3} M a^2 \ddot\theta - \frac{4}{3} M a^2 \dot\varphi^2 \sin\theta\cos\theta + Mg\, a \sin\theta = 0$

dividing by Ma $\Rightarrow$

$\frac{4}{3} a\ddot\theta - \frac{4}{3} a \dot\varphi^2 \sin\theta\cos\theta + g\sin\theta = 0$ (6)

which is the 1st eq^{n} of motion

Also substitution from (5) in the Lagrangian eq^{n}

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\varphi}\right) - \frac{\partial L}{\partial \varphi} = 0 \Rightarrow$

$\frac{d}{dt}\left(\frac{4}{3} M a^2 \dot\varphi \sin^2\theta\right) = 0$

$\therefore \quad \dot{\phi} \sin^2\theta = \text{Const.} = h \qquad (7)$

which is the 2nd eqtn of motion.

We the substitute for $\dot{\phi}$ from (7) in (6)

$\therefore \quad \frac{4}{3} a \ddot{\theta} - \frac{4}{3} a \frac{h^2}{\sin^3\theta} \cos\theta + g \sin\theta = 0 \qquad (8)$

Since ω

In order to determine the limits of motion we take

$\ddot{\theta} = \dot{\theta} \frac{d\dot{\theta}}{d\theta}$ in eqtn (8) and integrate w.r to θ.

This, however, is not required in the present problem.

* For the steady motion we take [$\theta = \alpha = \text{const.}$

$\dot{\phi} = \text{Const.} = \omega$] eqtn (9)

$\therefore \quad \dot{\theta} = \ddot{\theta} = 0 \quad , \quad \ddot{\phi} = 0 \qquad (10)$

Substituting in eqtn (6) we obtain

$\therefore \quad -\frac{4}{3} a \omega^2 \sin\alpha \cos\alpha + g \sin\alpha = 0$

If $\sin\alpha = 0$ (i.e. not in the vertical position)

$\Rightarrow \quad -\frac{4}{3} a \omega^2 \cos\alpha + g = 0 \Rightarrow \omega = \sqrt{3g/4a\cos\alpha} \qquad (i)$

The Condition of steady motion $\omega > 0$ (real) we,

therefore, $\cos\alpha > 0$ i.e $0 < \alpha < \pi/2 \qquad (11)$

Note: Condition (11) agrees with the general condition

$C^2 n^2 > 4 M g a A \cos\alpha'$ of the steady motion of the top

Since in this case $C = 0$, and the condition takes the form $-\cos\alpha' < 0$, $\pi/2 < \alpha' < \pi$

where α' is the angle between the axis of the top and the upward vertical $(\alpha = \pi - \alpha') \qquad (11)'$

In order to study small oscillations about the position of steady motion, we take $\theta = \alpha + \psi$ s.t $|\psi| \ll \alpha$.

Therefore, $\dot{\theta} = \dot{\psi}$, $\ddot{\theta} = \ddot{\psi}$, $\qquad (12)$

Since the position of initial steady motion ($\theta = \alpha$, $\psi = 0$) is a position of the motion then from eqn (7)

$$n = \omega \sin^2\alpha \quad (13)$$

Moreover, $\sin\theta = \sin(\alpha + \psi) = \sin\alpha\cos\psi + \cos\alpha\sin\psi$

ψ (small angle) $\Rightarrow \cos\psi \approx 1$, $\sin\psi \approx \psi \Rightarrow$

$$\sin\theta = \sin\alpha + \psi\cos\alpha = \sin\alpha(1 + \psi\cot\alpha)$$

Also, $\cos\theta = \cos(\alpha + \psi) = \cos\alpha\cos\psi - \sin\alpha\sin\psi =$

$$\cos\alpha - \sin\alpha\,\psi = \cos\alpha(1 - \psi\tan\alpha) \quad (14), b$$

We now substitute from (12, 13, 14 a, b) to find

$$\frac{4}{3}a\ddot{\psi} - \frac{4}{3}a\omega^2\sin^4\alpha\cdot\cos\alpha(1 - \psi\tan\alpha) =$$

$$\frac{4}{3}a\ddot{\psi} - \frac{4}{3}a\sin^3\alpha(1 + \psi\cot\alpha)^3 + g\sin\alpha(1 + \psi\cot\alpha)$$

(small quantity)

$$= 0$$

multiplying by $\frac{3}{4}$ $\Rightarrow$

$$a\ddot{\psi} - a\omega^2\sin^4\alpha\cos\alpha(1 - \psi\tan\alpha) -$$

$$\Rightarrow \quad \omega = \pm\sqrt{3g/4a\cos\alpha}$$

$$\ddot{\psi} - \omega\sin\alpha\cos\alpha(1 - \psi\tan\alpha)$$

Since $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2$

$n = -3$, $x = \psi\cot\alpha \Rightarrow$

$$\Rightarrow \ddot{\psi} - \omega^2\sin\alpha\cdot\cos\alpha(1 - \psi\tan\alpha)(1 - 3\psi\cot\alpha + \text{higher terms}) + \omega^2\cos\alpha\cdot\sin\alpha(1 + \psi\cot\alpha) = 0$$

$$\ddot{\psi} - \omega^2\sin\alpha\cdot\cos\alpha(1 - \psi\tan\alpha - 3\psi\cot\alpha + \cdots) + \omega^2\cos\alpha\sin\alpha(1 + \psi\cot\alpha) = 0$$

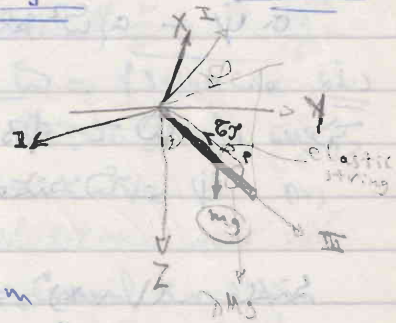
$$\Rightarrow \quad \ddot{\psi} + \omega^2\psi(\sin^2\alpha + 3\cos^2\alpha + \cos^2\alpha) = 0$$

$$\ddot{\psi} = -\psi\omega^2(1 + 3\cos^2\alpha)$$

which is a simple harmonic oscillation about the position of steady motion.

The angular freq. of the S.H. oscillation is $\omega\sqrt{1+3\cos^2\alpha}$ the Periodic time of the S.H.O is $T = \frac{2\pi}{\omega\sqrt{1+3\cos^2\alpha}}$

Ex.(3): A uniform rod OA for mass M and length 2a is free to rotate about its fixed end O a smooth ring of mass λM slides along the rod and is connected to O by an elastic string of modulus of elasticity $(n\,Mg)$, and natural length a. Apply Lagrange eqn's, to show that, if a steady motion is possible $\theta = \pi/3$, $\dot\varphi = \omega = \sqrt{3g/2a}$ then the length of the string $x = \frac{4}{3}a$ and $n = 1\lambda$



P (position of the ring) ; $OP = x$

γ (tension in the string)

$2a > x > a$

The generalized coordinates of the system are θ, φ, x the axes are identical with the example (1), (2)

Kinetic energy of the rod $= T_{rod} = \frac{2}{3}Ma^2(\dot\theta^2 + \dot\varphi^2\sin^2\theta)$ (1)

In order to find the K.E of the ring we note that

$\underline{r}_P = (0, 0, x)$, $\underline{v}_P = \dot{\underline{r}}_P + \underline{\Omega} \times \underline{r}_P$

$\underline{\Omega}$ angular velocity of the axes.

$\dot{\underline{r}}_P = (x\dot\theta, x\sin\theta\dot\varphi, \dot x) = \underline{v}_P$ (2)

$\Rightarrow T_{ring} = \frac{1}{2}\lambda M x\, v_P^2 = \frac{1}{2}\lambda M(\dot x^2 + x^2\sin^2\theta\cdot\dot\varphi^2 + x^2\dot\theta^2)$

Therefore, Total K.E $= T = (\frac{2}{3}Ma^2 + \frac{x^2}{2}\lambda M)(\dot\theta^2 + \dot\varphi^2\sin^2\theta) + \frac{1}{2}\lambda M\dot x^2$ (3)

The forces which can perform work are
Mg, λMg, $\mathcal{X}$, $\mathcal{Y}$ these forces are conservative
since the can be represented by potential energies

Potential energy due to gravity $= V_1 = -Mga\cos\theta - \lambda Mg X\cos\theta =$
$-Mg\cos\theta(a+\lambda X)$ (4) (A)
The string energy $= \frac{1}{2}K(l-l_0)^2 = V_2$
K = modulus of stiffness = (modulus of elasticity / l_0)
$\Rightarrow \frac{1}{2}(nMg/a)(X-a)^2 = V_2$ (5)
$\partial V_2/\partial X =$ tension $= \mathcal{Y} \rightarrow$ (6) $(X-a)^2 \frac{nMg}{a}(X)$
Total potential energy $= V = -Mg\cos\theta(a+\lambda X) + \frac{1}{2}\frac{nMg}{a}$
which is the potential energy of the system.

The Lagrangian function $L = T - V =$
$M(\frac{2}{3}a^2 + \frac{1}{2}\lambda X^2)(\dot\theta^2 + \dot\varphi^2\sin^2\theta) + \frac{1}{2}\lambda M\dot X^2 +$
$+ Mg\cos\theta(a+\lambda X) - \frac{1}{2}n\frac{Mg}{a}(X-a)^2$

The 1st Lagrangian eq^{tn}

$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) - \frac{\partial L}{\partial\theta} = 0$ (1st)

$\frac{\partial L}{\partial\dot\theta} = M(\frac{4}{3}a^2 + \lambda X^2)\dot\theta$,

$\frac{\partial L}{\partial\theta} = M(\frac{4}{3}a^2 + \lambda X^2)\dot\varphi^2\sin\theta\cos\theta$

$-Mg\sin\theta(a+\lambda X)$ $\Rightarrow$

$M(\frac{4}{3}a^2 + \lambda X^2)\ddot\theta + 2M\lambda X\dot X\dot\theta\left(-\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right)\right)$ (8)
$- M(\frac{4}{3}a^2 + \lambda X^2)\dot\varphi^2\sin\theta\cos\theta + Mg\sin\theta(a+\lambda X) = 0$
+

The 2nd Lagrangian eqtn

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - \frac{\partial L}{\partial \varphi} = 0$$

$$\frac{\partial L}{\partial \dot{\varphi}} = M\left(\frac{4}{3}a^2 + \lambda x^2\right)\dot{\varphi}\sin^2\theta \; ; \; \frac{\partial L}{\partial \varphi} = 0$$

$$M\left(\frac{4}{3}a^2 + \lambda x^2\right)\ddot{\theta} + 2M\lambda x\dot{x}\dot{\theta} - M\left(\frac{4}{3}a^2 + \lambda x^2\right)\dot{\varphi}^2 \sin\theta\cos\theta + Mg\sin\theta(a + \lambda x) = 0 \quad (8)$$

$$\left(\frac{4}{3}a^2 + \lambda x^2\right)\dot{\varphi}\sin^2\theta = \text{const} = h \quad (9)$$

the 3rd Lagrange eqtn

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$L = T - V = M\left(\frac{2}{3}a^2 + \frac{1}{2}\lambda x^2\right)\left(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta\right) + \frac{1}{2}M\dot{x}^2$$

$$\frac{\partial L}{\partial x} = \lambda M x\left(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta\right) + \lambda Mg\cos\theta - n\frac{Mg}{a}(x - a)$$

Therefore: $\lambda M\ddot{x} - \lambda M x\left(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta\right) - \lambda Mg\cos\theta + n\frac{Mg}{a}(x - a) = 0 \quad (10)$

For the steady motion, $\theta = \text{Const} = \alpha$; $\dot{\varphi} = \text{Const} = \omega$
ds from (9), $\Rightarrow$ therefore, x must be a Const $= l$
Therefore, $\dot{\theta} = \ddot{\theta} = 0$, $\dot{x} = \ddot{x} = 0$, $\ddot{\varphi} = 0 \quad (11)$
which are the conditions of steady motion.
Substituting in eqtn 8, 10 we find that

$$-\left(\frac{4}{3}a^2 + \lambda l^2\right)\omega^2\sin\alpha\cdot\cos\alpha + g\sin\alpha(a + \lambda l) = 0 \quad (12)$$

(if $\alpha \neq 0$) i.e vertical position $\Rightarrow$

$$-\lambda l\omega^2\sin^2\alpha - \lambda g\cos\alpha + \frac{ng}{a}(l - a) = 0 \quad (13)$$

eqtns (12), (13) are two eqtns in the unknowns

ω, α, l, n ; if $\omega = \sqrt{3g/2a}$, $\alpha = \pi/3$ then from these two eqtns we find

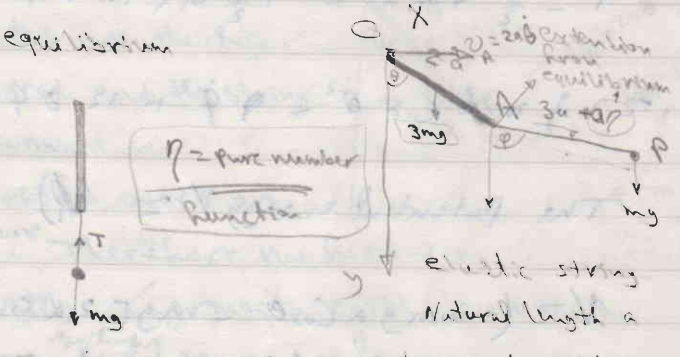
$$-\left(\frac{4}{3}a^2 + \lambda l^2\right)\cdot\frac{3g}{2a}\cdot\frac{1}{2} + g(a + \lambda l) = 0 \quad (12')$$

$$-a - 3\lambda/4a + a + \lambda l = 0 \Rightarrow \lambda = (4a/3)$$

From (13), $-\lambda\frac{4}{3}a\cdot\frac{3g}{2a}\cdot\frac{3}{4} - \lambda g\left(\frac{1}{2}\right) + (ng/a)\cdot(a/3) = 0$

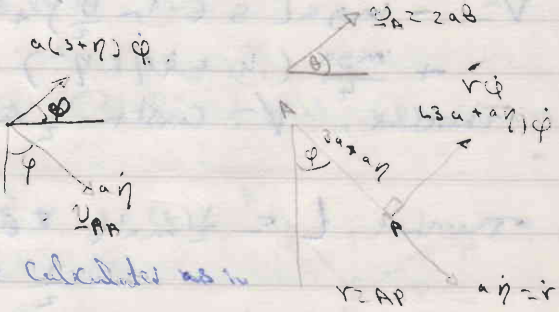
$$\frac{n}{3} = 2\lambda \Rightarrow n = 6\lambda$$

Generalized Coordinate



1. The generalized coordinates are θ, φ, η
   The system is conservative

2. K.E $= T = T_{rod} + T_{particle} = \frac{1}{2}\frac{M}{3}v^2 + \frac{1}{2}m_p v_p^2$
   $= \frac{1}{2}\frac{M}{3}(3m)a^2\dot{\theta}^2 + \frac{1}{2}m v_p^2$
   $v_p = v_A + v_{PA}$



* As an alternative method can we calculate as in the following way.

$\underline{r}_p = [x_p, y_p] = [\,2a\sin\theta + a(3+\eta)\sin\varphi\,,\; 2a\cos\theta + a(3+\eta)\cos\varphi\,]$

$\therefore \quad \underline{v}_p = \dot{\underline{r}}_p = [\,2a\cos\theta\cdot\dot\theta + a(3+\eta)\cos\varphi\cdot\dot\varphi + a\dot\eta\sin\varphi\,,$

$-2a\sin\theta\cdot\dot\theta - a(3+\eta)\sin\varphi\,\dot\varphi + a\dot\eta\cos\varphi\,]$

For small oscillation about the position of equilibrium

θ, φ, η , $\dot\theta, \dot\varphi, \dot\eta$, $\theta, \ddot\varphi, \ddot\eta$ are small quantities

at position of equi $(\theta, \varphi, \eta) = (0, 0, 0)$

Therefore, $T = \frac{1}{2} m v_p^2$ $\Rightarrow$ taking 2nd order approximation

since T is 2nd order

Therefore $\underline{v}_p = [\,2a\dot\theta + 3a\dot\varphi\,,\; a\dot\eta\,]$

$\therefore \quad T = \frac{1}{2}\,4ma^2\dot\theta^2 + \frac{1}{2}m(4a^2\dot\theta^2 + 9a^2\dot\varphi^2 + 12a^2\dot\theta\dot\varphi + a^2\dot\eta^2)$

$= \frac{1}{2}ma^2(8\dot\theta^2 + 9\dot\varphi^2 + 12\dot\theta\dot\varphi + \dot\eta^2)$

The potential energy $V = V_1)_{\text{gravity}} + V_2)_{\text{strain energy}}$

$V = (-3mga\cos\theta - mg[\,2a\cos\theta + a(3+\eta)\cos\varphi\,]) +$

$\frac{1}{2}K(3a + \eta - a)^2 =$ (a: natural length)

$-mga(3\cos\theta + 3 + \eta)\cos\varphi) + \frac{1}{2}\frac{mga^2}{2a}(2+\eta)^2 \Rightarrow$

$\therefore$ (2nd order expansion))

$V = -mga[\,5(\underset{\text{Const.}}{1} - \frac{\theta^2}{2}) + 3(\underset{\text{Const.}}{1} - \frac{\varphi^2}{2}) + \eta(\underset{\text{Const.}}{1} - \frac{\varphi^2}{2})\,]$ (1st order; 3rd order; Const.)

$+ \frac{mga}{4}(4 + 4\eta + \eta^2) =$

Therefore, $V = \text{Const} + \frac{1}{2}mga(5\theta^2 + 3\varphi^2) + \frac{mga}{4}\eta^2$

Therefore $L = \frac{1}{2}ma^2(8\dot\theta^2 + 9\dot\varphi^2 + 12\dot\theta\dot\varphi + \dot\eta^2) -$

$-\frac{1}{2}mga(5\theta^2 + 3\varphi^2) - \frac{mga}{4}\eta^2$

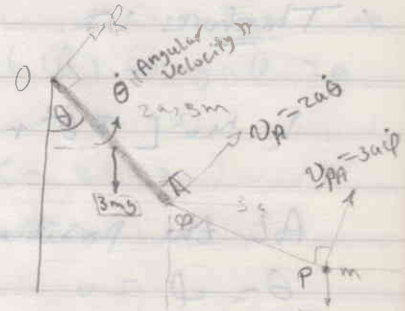
The two Lagrangian eqtns

$\frac{d}{dt}\frac{\partial L}{\partial\dot\theta} - \frac{\partial L}{\partial\theta} = 0$, $\frac{d}{dt}\frac{\partial L}{\partial\dot\varphi} - \frac{\partial L}{\partial\varphi} = 0$

will give the same results as in example [1]

A. Mech. The Theory of Small oscillations about the position of Stable equilibrium.

Example: A uniform rod OA of mass (3m) and length 2a swings about a smooth hinge at O. A particle of mass m is attached to A by inelastic string of length 3a. If the motion is restricted in the vertical plane of the rod. Apply Lagrange's eqtns to study the small oscillations about the position of equilibrium. Find then the normal frequencies, the normal periods, the equivalent lengths of simple pendulum, and the normal modes vibrations.



The generalized Coordinates are the two angles θ, φ which the rod and the string make (respectively) with the vertical downward. The forces which can perform work are the weight $3mg$, mg. Therefore the field is Conserved. In order to apply Lagrange's eqtns we have to determine T, V, L (all K. Ener.; P.E. energ., Lagrangian)

$$T = T)_{rod} + T)_{Particle}$$

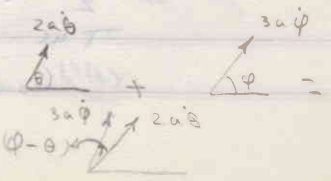
$\dot{\theta}$ is the angular velocity of the rod $\Rightarrow$

$$T = \frac{1}{2} I_O \dot{\theta}^2$$ where O fixed point. (plane motion)

$$\frac{1}{2} \cdot \frac{4}{3} (3m)\, a^2 \dot{\theta}^2 = \frac{1}{2} (4ma^2 \dot{\theta}^2) \qquad (1)$$

$$T)_{Particle} = \frac{1}{2} m_p v_p^2$$

But $\underline{v}_p = \underline{v}_A + \underline{v}_{p \to A}$ (relative)



$$v_p^2 = 4a^2\dot\theta^2 + 9a^2\dot\varphi^2 + 2(2a\dot\theta)(3a\dot\varphi)\cos(\varphi-\theta)$$

$$T_{particle} = \tfrac{1}{2}m\left[4a^2\dot\theta^2 + 9a^2\dot\varphi^2 + 12a^2\dot\varphi\dot\theta\cos(\varphi-\theta)\right] \quad (2)$$

$$T = \tfrac{1}{2}ma^2\left[8\dot\theta^2 + 9\dot\varphi^2 + 12\dot\theta\dot\varphi\cos(\varphi-\theta)\right] \quad (3)$$

* Potential energy $= V = -3mga\cos\theta - mg(2a\cos\theta + 3a\cos\varphi) =$
$-mga(5\cos\theta + 3\cos\varphi)$

* Therefore: $L = T - V = \tfrac{1}{2}ma^2$

$$\left[8\dot\theta^2 + 9\dot\varphi^2 + 12\dot\theta\dot\varphi\cos(\varphi-\theta)\right] + mga(5\cos\theta + 3\cos\varphi) \quad (5)$$

At the position of stable equilibrium.
$\theta = \varphi = 0$, & and the rod and the string are along the vertical & downward.

$$\dot\theta = \ddot\theta = \dot\varphi = \ddot\varphi = 0 \quad (6)$$ "position of stable equilibrium"

For small oscillations about this position $\theta, \varphi, \dot\theta, \dot\varphi, \ddot\theta, \ddot\varphi$ are small quantities.

Therefore, $\sin\theta = \theta - \theta^3/3!$

$\cos\theta = 1 - \theta^2/2!$ "2nd order terms only" (7)

Substit. in eq^{n} (5)" we find that

$$L = \tfrac{1}{2}ma^2\left[8\dot\theta^2 + 9\dot\varphi^2 + 12\dot\theta\dot\varphi(1)\right] + mga\left[5(1-\theta^2/2) + 3(1-\varphi^2/2)\right]$$

" Constant terms can be dropped "

$$\Rightarrow L = \text{Constant} + \tfrac{1}{2}ma^2(8\dot\theta^2 + 9\dot\varphi^2 + 12\dot\theta\dot\varphi) - \frac{mga}{2}(5\theta^2 + 3\varphi^2) \quad (8)$$

The Lagrangian of small oscillations about the position of stable equilibrium.

Consequently,

$$\frac{\partial L}{\partial \dot\theta} = ma^2(8\dot\theta^2 + 6\dot\varphi)\;;$$

$$\frac{\partial L}{\partial \theta} = -mga(5\theta)$$

$$\frac{\partial L}{\partial \dot\varphi} = ma^2(3\dot\varphi + 6\dot\theta)\;,\quad \frac{\partial L}{\partial \varphi} = -mga(3\varphi) \qquad (9)$$

* The 1st Lagrangian equation.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} - \frac{\partial L}{\partial \theta} = 0 = ma^2(8\ddot\theta + 6\ddot\varphi) + 5mga\theta = 0$$

$$\therefore\; 8\ddot\theta + 6\ddot\varphi + 5K\theta = 0 \qquad (10)\quad (K = g/a)$$

* The 2nd Lagrangian equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\varphi} - \frac{\partial L}{\partial \varphi} = 0$$

$$ma^2(3\ddot\varphi + 6\ddot\theta) + 3mga\varphi = 0$$

$$2\ddot\theta + 3\ddot\varphi + K\varphi = 0 \qquad (11)$$

"2.5.H oscillations grouped together"

Solution:

$A\cos(\omega t + \epsilon)$

$A\,e^{i(\omega t+\epsilon)}$

Seeking for a periodic solution we may take $\theta = A\cos(\omega t+\epsilon)$ or $\theta = A e^{i(\omega t+\epsilon)}$ and $\varphi = B\cos(\omega t+\epsilon)$ or $\varphi = Be^{i(\omega t+\epsilon)}$ (12)

Substution of (12) in (10), (11) gives

$$-8\omega^2 A\cos(\omega t+\epsilon) - 6\omega^2 B\cos(\omega t+\epsilon) + 5KA\cos(\omega t+\epsilon) = 0$$

$$(5K - 8\omega^2)A - 6\omega^2 B = 0 \qquad (13)$$ eigenvalue problem

$$-2\omega^2 A - 3\omega^2 B + KB = 0$$

$$-2\omega^2 A + B(K - 3\omega^2) = 0 \qquad (14)$$

(13) , (14) are two linear homogeneous equations in A, B

For non-trivial solution the determinant of the coefficients must vanish, and accordingly

$$\begin{vmatrix} 5K - 8\omega^2 & -6\omega^2 \\ -2\omega^2 & K - 3\omega^2 \end{vmatrix} = 0$$

$$(5K - 8\omega^2)(K - 3\omega^2) - 12\omega^4 = 0$$

$$12\omega^4 - 23\omega^2 K + 5K^2 = 0$$

$$12(\omega^2)^2 - 23K(\omega^2) + 5K^2 = 0 \Rightarrow (4\omega^2 - K)(3\omega^2 - 5K) = 0$$

$$\omega^2 = \frac{23K \pm \sqrt{(23K)^2 - 4 \times 5K^2 \times 12}}{2 \times 12}$$

$\omega^2 = 9/4K = 9/4a$; $\omega^2 = 5K/3 = 5g/3a$

"which are normal angular frequencies"

Therefore the normal angular frequencies are.

$\omega_1 = \sqrt{9/4a} = \frac{1}{2}\sqrt{9/a}$,

* $\omega_2 = \sqrt{5g/3a}$ => The normal frequencies

$f_1 = \omega_1/2\pi = \frac{1}{4\pi}\sqrt{9/a}$

$f_2 = \omega_2/2\pi = \frac{1}{2\pi}\sqrt{5g/3a}$

* The normal periodic times $T_1 = \frac{2\pi}{\omega_1} = 2\pi\sqrt{\frac{4a}{g}} = 4\pi\sqrt{\frac{a}{g}}$.

$T_2 = \frac{2\pi}{\omega_2} = 2\pi\sqrt{\frac{3a}{5a}}$.

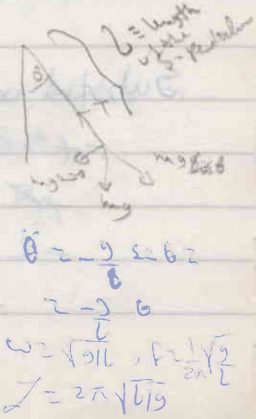
In order to find the equivalent lengths of simple pendulum.

we note that for a simple pendulum

Therefore, $l_1 = 4a$; $l_2 = 3a/5$.

*Finelly to find the normal modes of vibrations

we 1st take $\omega^2 = \omega_1^2 = K/4$ in eqn (14)



$\ddot{\theta} = -\frac{g}{l}\sin\theta$

$= -\frac{g}{l}\theta$

$\omega = \sqrt{g/l}$, $f = \frac{1}{2\pi}\sqrt{\frac{g}{l}}$

$T = 2\pi\sqrt{l/g}$

$$-2\left(\frac{K}{4}\right)A_1 + B_1\left(K - \frac{3K}{4}\right) = 0$$

$$-\frac{A_1}{2} + \frac{B_1}{4} = 0 \Rightarrow B_1 = 2A_1$$

$$[A_1 \; B_1]\begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = 1$$

Therefore, $\theta = A_1 \cos(\omega_1 t + \epsilon_1)$

$\phi = 2A_1 \cos(\omega_1 t + \epsilon_1)$

$$\begin{bmatrix} \theta \\ \phi \end{bmatrix} = A_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cos(\omega_1 t + \epsilon_1)$$ [1st mode of vibration]

In this mode, the rod and the string vibrate in the same direction. (w.r.t. the vertical downward)

For the 2nd mode of vibration, we take.

$\omega^2 = \omega_2^2 = 5K/3$ = in eqn (14)

Therefore, $-2\left(\frac{5K}{3}\right)A_2 + B_2(K - 5K) = 0$

$-\frac{10}{3}A_2 - 4B_2 = 0 \Rightarrow 5A_2 = -6B_2$

i.e. $A_2 : B_2 = 6 : -5$

Therefore $A = 6A_2' \Rightarrow B_2 = -5A_2'$

$\theta = 6A_2 \cos(\omega_2 t + \epsilon_2)$; $\phi = -5A_2' \cos(\omega_2 t + \epsilon_2)$.

$$\begin{bmatrix} \theta \\ \phi \end{bmatrix} = A_2 \begin{bmatrix} 6 \\ -5 \end{bmatrix} \cos(\omega_2 t + \epsilon_2)$$

which is the 2nd normal mode of vibration.

In this mode the rod and the string vibrate in opposite directions w.r.t. to the vertical downward.

The general solution of eqtns (1) (Periodic Solution) takes the form $\begin{bmatrix}\theta\\ \varphi\end{bmatrix} = A_1 \begin{bmatrix}1\\ 2\end{bmatrix} \cos(\omega_1 t + \epsilon_1) + A_2 \begin{bmatrix}6\\ -5\end{bmatrix} \cos(\omega_2 t + \epsilon_2)$

1st (V. Imp) for Physics and Solid State Physics.

Examples: In the above example, if the string is elastic with natural length a, and bob is in the position of equilibrium Find the normal modes of vibrations.

Note: $L = \frac{1}{2}ma^2[8\dot\theta^2 + 3\dot\varphi^2 + 12\dot\theta\dot\varphi\cos(\varphi-\theta)] + mga(5\cos\theta + 3\cos\varphi)$ (3)

The general eqtns of motion can be obtained from eqtn (3) in the following way

$\frac{\partial L}{\partial \dot\theta} = ma^2[8\dot\theta + 6\dot\varphi\cos(\varphi-\theta)]$

$\frac{\partial L}{\partial \theta} = ma^2[6\dot\theta\dot\varphi + \sin(\varphi-\theta)] = 5mga\sin\theta.$

Therefore, the 1st Lagrangian eqtn

1st $\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} - \frac{\partial L}{\partial \theta} = 0$

$8\ddot\theta + 6\ddot\varphi\cos(\varphi-\theta) + 6\dot\varphi\sin(\varphi-\theta)(\dot\varphi-\dot\theta) = -6\dot\theta\dot\varphi\sin(\varphi-\theta) + 5K\sin\theta = 0$

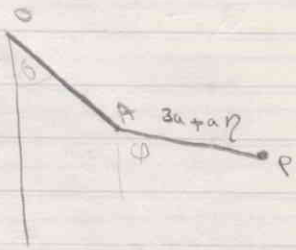
$\therefore 8\ddot\theta + 6\ddot\varphi\cos(\varphi-\theta) - 6\dot\varphi^2\sin(\varphi-\theta) + 5K\sin\theta = 0$

(which reduces to that of Small oscillations)

2nd $\frac{d}{dt}\frac{\partial L}{\partial \dot\varphi} - \frac{\partial L}{\partial \varphi} = 0$

$AP = 3a + a\eta$

$a\eta$ measures the extention from the position of equilibrium.



At the position of equilibrium

$$mg = K(3a - a) \quad ; K = \frac{mg}{2a}.$$

The 3rd Lagrangian eqtn gives

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\eta} - \frac{\partial L}{\partial \eta} = 0$$

$$\frac{\partial L}{\partial \dot\eta} = ma^2\dot\eta \,, \quad \frac{\partial L}{\partial \eta} = -\frac{mga}{2}\eta$$

Therefore, $ma^2\ddot\eta + \frac{mga}{2}\eta = 0$

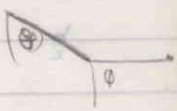
$\therefore \ddot\eta = -\frac{K}{2}\eta$ which is a S.H. oscillation of angular frequency

$\omega_3 = \sqrt{\frac{K}{2}} = \sqrt{\frac{g}{2a}}$. The equivalent length of simple pendulum of this oscillation is $(2a)$.

Also $\eta = A_3 \cos(\omega_3 t + \epsilon_3)$

* The 3 normal modes of vibrations are

(I) $\begin{bmatrix}\theta\\ \varphi\end{bmatrix} = A\begin{bmatrix}1\\ 2\end{bmatrix}\cos(\omega_1 t + \epsilon_1)$ From example (1)



(II) $\begin{bmatrix}\theta\\ \varphi\end{bmatrix} = A_2\begin{bmatrix}6\\ -5\end{bmatrix}\cos(\omega_2 t + \epsilon_2)$

$\eta = A_3\cos(\omega_3 t + \epsilon_3)$

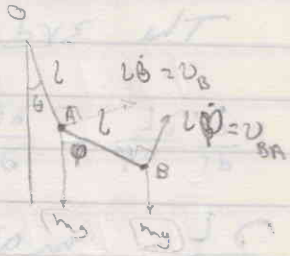
Then, the general solution of the problem, may also be written in the form

$$\begin{bmatrix}\theta\\ \varphi\\ \eta\end{bmatrix} = A_1\begin{bmatrix}1\\ 2\\ 0\end{bmatrix}\cos(\omega_1 t + \epsilon_1) + A_2\begin{bmatrix}6\\ -5\\ 0\end{bmatrix}\cos(\omega_2 t + \epsilon_2)$$

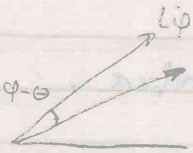
$$+ A_3\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}\cos(\omega_3 t + \epsilon_3) \,.$$

* Double Pendulum:

$$T = T)_A + T)_B = \frac{1}{2} m v_A^2 + \frac{1}{2} m v_B^2$$



* $v_B = v_A + v_{BA}$

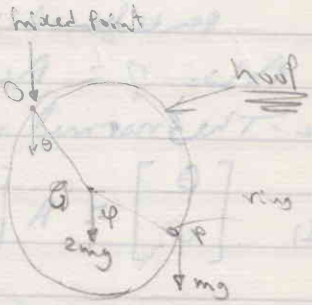


$$T = \frac{1}{2} m l^2 \dot\theta^2 + \frac{1}{2} m l^2 [\dot\varphi^2 + \dot\theta^2 + 2\dot\theta\dot\varphi \cos(\varphi - \theta)]$$

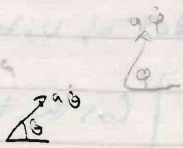
$$V = V - mgl\cos\theta - mgL(\cos\theta + \cos\varphi)$$
$$= -mgl(2\cos\theta + \cos\varphi)$$

and Complete as previously learnt

* $T = T)_{hoop} + T)_{ring} =$



$\frac{1}{2} I_O \dot\theta^2 + \frac{1}{2} m_P v_P^2$ $v_P = v_G + v_{PG}$



$I_G = M_{hoop} a^2$

$I_O = I_G + M_{hoop} a^2 = 4 M a^2$

The hoop vibrating about O

$$\Rightarrow T = \frac{1}{2} 4 m a^2 \dot\theta^2 + \frac{1}{2} m a^2 [\dot\theta^2 + \dot\varphi^2 + 2\dot\theta\dot\varphi\cos(\varphi - \theta)] =$$

$$\frac{1}{2} m a^2 [5\dot\theta^2 + \dot\varphi^2 + 2\dot\theta\dot\varphi\cos(\varphi - \theta)]$$

* $V = -2mga\cos\theta - mg(a\cos\theta + a\cos\varphi) =$
$-mga(3\cos\theta + \cos\varphi)$

$$L)_{S.Osc} = T - V = \frac{1}{2} m a^2 [5\dot\theta^2 + \dot\varphi^2 + 2\dot\theta\dot\varphi^2] - \frac{1}{2} mag[3\theta^2 + \varphi^2]$$

→ Lagrange eqt's etc

the final eqns become

$$5\ddot{\theta} + \ddot{\varphi} + 3K\theta = 0 \qquad \theta = A\cos(\omega t + \epsilon)$$

$$\ddot{\varphi} + \ddot{\theta} + K\varphi = 0 \qquad \varphi = B\cos(\omega t + \epsilon)$$

Eigenvalue Problem.

$$(5K - 5\omega^2)A - \omega^2 B = 0$$

$$-\omega^2 A + (K - \omega^2)B = 0 \Rightarrow$$

$$\begin{vmatrix} \quad & \quad \\ \quad & \quad \end{vmatrix} = 0$$

$$\begin{cases} 3K^2 - 8K\omega^2 + 4\omega^4 = 0 \\ (3\omega^2 - 3K)(2\omega^2 - K) = 0 \end{cases}$$

$\omega_2^2 = \frac{3K}{2}$, $\omega_1^2 = \frac{K}{2}$; ——

for ω_1^2 ; $A_1 = B_1$ $\Rightarrow$ in this mode

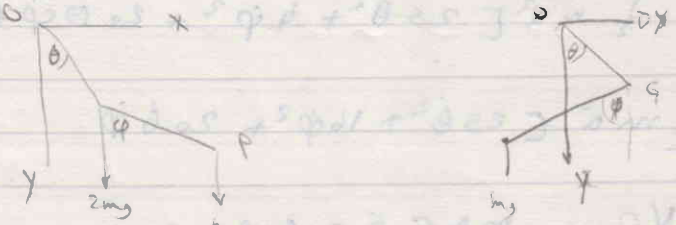
$\theta = A_1\cos(\omega_1 t + \epsilon_1)$, $\varphi = A_1\cos(\omega_1 t + \epsilon_1)$

i.e the ring is at rest w.r to the hoop

for ω_2^2 ; $\Rightarrow B_2 = -3A_2$

$$\begin{bmatrix} \theta \\ \varphi \end{bmatrix} = A_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} \cos(\omega_2 t + \epsilon_2)$$

To prove that the Center of gravity of system and hoop always moves on the y axis vertical line



$$3m\bar{x} = 2ma\sin\theta + m(a\sin\theta + a\sin\varphi) =$$

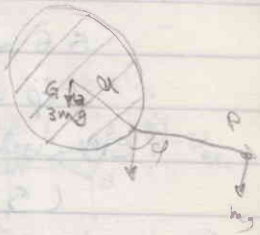
$$ma(3\sin\theta + \sin\varphi) \approx ma(3\theta + \varphi) \quad \text{for small oscillation}$$

$$\Rightarrow ma(3A_2 + B_2)\cos(\omega_2 t + \epsilon_2) = 0$$

$$\frac{K}{2}B_2 = 0$$

* $T = \frac{1}{2} I_G \dot{\theta}^2 + \frac{1}{2} m_p v_p^2$

Disc Particle



$I_G = \frac{1}{2} M_{disc} a^2 = \frac{1}{2} 3 m a^2$

$T = \frac{1}{2} \frac{3}{2} m a^2 \dot{\theta}^2 + \frac{1}{2} m a^2 [\dot{\theta}^2 + \dot{\varphi}^2 + 2\dot{\theta}\dot{\varphi}\cos(\varphi - \theta)] =$

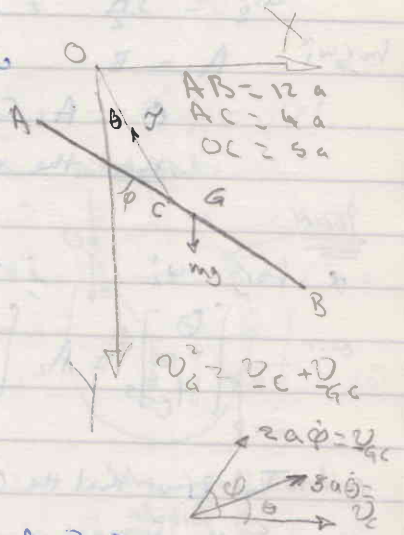
$\frac{1}{2} m a^2 [\frac{5}{2}\dot{\theta}^2 + \dot{\varphi}^2 + 2\dot{\theta}\dot{\varphi}\cos(\varphi - \theta)$

* $V = -mga(\frac{1}{2}\cos\theta + \cos\theta)$

A uniform rod AB

Work is only done by mg the conservative

There is no fixed point in the rod



$T = T_{rod} = \frac{1}{2} I_G \dot{\varphi}^2 + \frac{1}{2} m v_G^2 =$

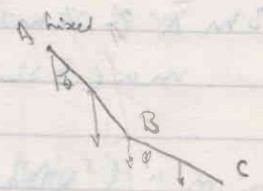
$\frac{1}{2} \cdot \frac{1}{3} m (6a)^2 \dot{\varphi}^2 + \frac{1}{2} m a_G^2$

$\frac{1}{2} m a^2 [25\dot{\theta}^2 + 4\dot{\varphi}^2 + 20\dot{\theta}\dot{\varphi}\cos(\varphi - \theta)] =$

$\frac{1}{2} m a^2 [25\dot{\theta}^2 + 16\dot{\varphi}^2 + 20\dot{\theta}\dot{\varphi}$

$V = -mg[5a\cos\theta + 2a\cos\varphi]$

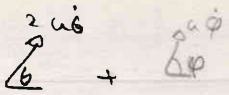
* Two rods



$T = T)_{AB} + T)_{BC} \Rightarrow$

$T)_{AB} = \frac{1}{2} I_A \dot{\theta}^2 = \frac{1}{2} 2m \frac{4}{3} a^2 \dot{\theta}^2 = \frac{1}{2} \frac{8}{3} m a^2 \dot{\theta}^2$

$$T_{BC} = \frac{1}{2} I_G \dot{\varphi}^2 + \frac{1}{2} M_{BC} v_G^2$$



$$\frac{1}{2}\left(\frac{3a}{3}\right) a^2 \dot{\varphi}^2 + \frac{1}{2} 3m a^2 \left(4\dot{\theta}^2 + \dot{\varphi}^2 + 4\dot{\theta}\dot{\varphi}\cos(\varphi-\theta)\right) \Rightarrow$$

$$T = \frac{1}{2} m a^2 \left[\left(\frac{44}{3}\right)\dot{\theta}^2 + 4\dot{\varphi}^2 + 12\dot{\theta}\dot{\varphi}\cos(\varphi-\theta)\right]$$

$$V = -2mg\,a\cos\theta - 3mg\left[2a\cos\theta + a\cos\varphi\right].$$

A. Mech. Hamilton Equations:

The Poiss

3) $\{q_j, h\} = \frac{\partial h}{\partial p_j}$; $\{p_j, h\} = -\frac{\partial h}{\partial q_j}$

$$\{q_j, h\} = \sum_i \left(\frac{\partial q_j}{\partial q_i} \frac{\partial h}{\partial p_i} - \frac{\partial q_j}{\partial p_i} \frac{\partial h}{\partial q_i} \right)$$

($\frac{\partial q_j}{\partial q_i} = \delta_{ij}$, $\frac{\partial q_j}{\partial p_i} = 0$)

$$\{p_j, h\} = \sum_i \left(\frac{\partial p_j}{\partial q_i} \frac{\partial h}{\partial p_i} - \frac{\partial p_j}{\partial p_i} \frac{\partial h}{\partial q_i} \right) = -\frac{\partial h}{\partial q_j}$$

4) If $\underline{h}$ is the angular momentum of a particle about the fixed origin O, prove that $\{h_x, y\} = z$

$\{h_x, p_y\} = p_z$; $\{h_x, h_y\} = h_z$

Proof:

$$\underline{h} \times \underline{p} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix} = y p_z - z p_y$$

$$(y p_z - z p_y)\underline{i} + (z p_x - x p_z)\underline{j} + (x p_y - y p_x)\underline{k}$$

$$h_x \underline{i} + h_y \underline{j} + h_z \underline{k}$$

$$\{h_x, y\} = \frac{\partial h_x}{\partial x}\cdot\frac{\partial y}{\partial p_x} + \frac{\partial h_x}{\partial y}\cdot\frac{\partial y}{\partial p_y} + \frac{\partial h_x}{\partial z} + \frac{\partial h_x}{\partial p_z}\frac{\partial y}{\partial z} -$$

$$- \frac{\partial h_x}{\partial p_x}\cdot\frac{\partial x}{\partial x} - \frac{\partial h_x}{\partial p_y}\frac{\partial y}{\partial y} - \frac{\partial h_x}{\partial p_z}\cdot\frac{\partial z}{\partial z} = -\frac{\partial h_x}{\partial p_y} = -(-z) = z$$

$$\{h_x, p_y\} = \frac{\partial h_x}{\partial x}\cdot\frac{\partial p_y}{\partial p_x} + \frac{\partial h_x}{\partial y}\frac{\partial p_y}{\partial p_y} + \frac{\partial h_x}{\partial z}\frac{\partial p_y}{\partial p_z} -$$

$$- \frac{\partial h_x}{\partial p_x}\frac{\partial p_y}{\partial x} - \frac{\partial h_x}{\partial p_y}\frac{\partial p_y}{\partial y} - \frac{\partial h_x}{\partial p_z}\frac{\partial p_y}{\partial z} =$$

$$\{h_x, h_y\} = \frac{\partial h_x}{\partial x}\cdot\frac{\partial h_y}{\partial p_x} + \frac{\partial h_x}{\partial y}\cdot\frac{\partial h_y}{\partial p_y} + \frac{\partial h_x}{\partial z}\cdot\frac{\partial h_y}{\partial p_z}$$

$$- \frac{\partial h_x}{\partial p_x}\cdot\frac{\partial h_y}{\partial x} - \frac{\partial h_x}{\partial p_y}\frac{\partial h_y}{\partial y} - \frac{\partial h_x}{\partial p_z}\cdot\frac{\partial h_y}{\partial z} =$$

h_x, p_x, x

$$= \frac{\partial h_x}{\partial z}\frac{\partial h_y}{\partial p_z} - \frac{\partial h_x}{\partial p_z}\frac{\partial h_y}{\partial z} = -(p_y)(-x) - y p_x =$$

$$x p_y - y p_x = h_z = \{h_x, h_y\}$$ « Poisson bracket. »

* The Hamiltonian of a holonomic conservative system :

For a holonomic conservative system of n generalized coordinates $(q_i, i = 1, 2, \ldots n)$ the Hamiltonian H is defined by

$$H = -L + \sum_{i=1}^{n} p_i \dot{q}_i \quad [1]$$ where $L(q_i, \dot{q}_i, t)$ is the Lagrangian function of the system. p_i, $i = 1, 2, \ldots n$ are the generalized components of momentum being defined by

$$p_i = \frac{\partial T}{\partial \dot{q}_i} \overset{*}{=} \frac{\partial L}{\partial \dot{q}_i} \quad [2]$$ * since $L = T - V(q_1 \ldots q_n)$ where V is dependent of $\dot{q}_i$

« which gives n linear eqtns of in terms of p_i »

$H(q_i, p_i, t)$ — Compare $L(q_i, \dot{q}_i, t)$

eqtn. [2] is a group of (n) linear eqtns of p_i in terms of $\dot{q}_i$, $(i = 1, 2, \ldots, n)$.

(Note: that T is quadratic in $\dot{q}_i$ and therefore $\frac{\partial T}{\partial \dot{q}_i}$ is a linear in $\dot{q}_i$)

These eqtns can thus be solved to find $\dot{q}_i$ in terms of p_i $(i = 1, 2, \ldots, n)$.

Consequently the Hamiltonian H can be regarded as a function of q_i, p_i, t $(i = 1, 2, \ldots n)$

i.e $H \equiv H(q_i, p_i, t)$.

Hamilton's Equations.

For a holonomic Conservative system Hamilton's eqtns are given by $\dot{q}_i = \frac{\partial H}{\partial p_i}$; $\dot{p}_i = -\frac{\partial H}{\partial q_i}$, $i = 1, 2, \ldots n$.

Hamiltons Eqtns [5]

which are $2n$ linear differential equations.

Equation [3] is a group of $2n$ DE's of the 1st order.

On the other hand Lagrange's eqtns are n DE's of the 2nd order.

Hamilton's eqtns [5] Can be obtained either by using Lagrange's eqtns or Hamilton's Principle.

+ Derivations using Lagrange's Equations:

Since $H = H(q_i, p_i, t) \rightarrow$

$$dH = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i \right) + \frac{\partial H}{\partial t} dt \qquad [4]$$

Also from [1] $H = -L(q_i, \dot{q}_i, t) + \sum_i p_i \dot{q}_i$

Therefore

$$dH = -\sum_i \left(\frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i \right) - \frac{\partial L}{\partial t} dt + \sum_i (p_i d\dot{q}_i + \dot{q}_i dp_i) \qquad [5]$$

But, $p_i = \frac{\partial L}{\partial \dot{q}_i}$ from [2]

And from Lagrange's Eq'n

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \quad \Rightarrow \quad \frac{d}{dt}\dot{p}_i = \frac{\partial L}{\partial q_i} \qquad (6)$$

$\frac{\partial L}{\partial \dot{q}_i} = p_i$ Substituting in (5) we find

$$dH = -\sum (\dot{p}_i\, dq_i + p_i\, d\dot{q}_i) - \frac{\partial L}{\partial t}dt + \sum_i (p_i\, d\dot{q}_i + \dot{q}_i\, dp_i)$$

$$\Rightarrow dH = \sum_i (-\dot{p}_i\, dq_i + \dot{q}_i\, dp_i) - \frac{\partial L}{\partial t}dt \qquad (7)$$

Comparing (7) & (4) we find that ; Since dq_i, dp_i, dt are linearly independent

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \quad ; \quad \dot{q}_i = \frac{\partial H}{\partial p_i}, \quad i = 1, 2, \ldots n$$

Hamilton's eqtns

Also $\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$ (8)

* Derivations by using Hamilton's Principle :

$$S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)\, dt = \int_{t_1}^{t_2} [-H(q_i, p_i, t) + \sum_i p_i \dot{q}_i]\, dt$$

$$= \int_{t_1}^{t_2} f(q_i, p_i, \dot{q}_i, \dot{p}_i, t)\, dt \qquad (9)$$

where $f(q_i, p_i, \dot{q}_i, t) = \sum_i p_i \dot{q}_i - H(q_i, p_i, t)$ (10)

In Eqtns (9), (10), f is regarded as a function of $4n+1$ parameter. The n components of Coordinates $(q_i,\ i = 1, 2, \ldots n)$, the n components of momentum $(p_i,\ i = 1, 2, \ldots n)$, and the $2n$ derivatives $(\dot{q}_i, \dot{p}_i,\ i = 1, 2, \ldots n)$, and the time t

According to Hamiltons principle the actual path of motion is determined when S is extremum ; Also according to Euler - Lagrange Differential Principle

S is extremum when

$$\frac{d}{dx}\frac{\partial h}{\partial y'_i} - \frac{\partial h}{\partial y} = 0$$

$$\therefore \frac{d}{dt}\frac{\partial h}{\partial \dot{q}_i} - \frac{\partial h}{\partial q_i} = 0 \qquad (i = 1, 2, \ldots, n) \qquad (11a)$$

and $$\frac{d}{dt}\frac{\partial h}{\partial \dot{p}_i} - \frac{\partial h}{\partial p_i} = 0 \qquad (i = 1, 2, \ldots n) \qquad (11b)$$

$(11a) \Rightarrow \frac{d}{dt} p_i - \left(-\frac{\partial H}{\partial q_i}\right) = 0$

$$\therefore \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad i = 1, 2, \ldots n$$

$$\frac{d}{dt}(0) - \left(\dot{q}_i - \frac{\partial H}{\partial p_i}\right) = 0$$

Therefore $$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad i = 1, 2, \ldots n$$

which are "Hamilton's Equations"

Eg: Consider the plane motion of a particle under the action of a Central force of potential $V(r)$

* The generalized Coordinates are (r, θ)
* The Kinetic energy $T = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2)$
* The Lagrangian $L = T - V = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$
* The generalized Components of momentum are :

$$p_r = \frac{\partial T}{\partial \dot{r}} = m\dot{r} \; ; \; p_\theta = mr^2\dot{\theta}$$

$$\dot{r} = \frac{p_r}{m} \; ; \; \dot{\theta} = \frac{p_\theta}{mr^2}$$

The Hamiltonian $H = -L + \sum_i p_i \dot{q}_i = -L + p_r \dot{r} + p_\theta \dot{\theta} =$
$-L + m\dot{r}^2 + mr^2\dot{\theta}^2 = V + T = H$
$\quad\quad\quad

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}$$

$\therefore H = \text{Constant}$ (Constant of the motion)

Note (1)

$$P_s = \frac{\partial T}{\partial \dot{q}_s}, \quad T = \frac{1}{2}\sum_i m_i \dot{\underline{r}}_i \cdot \dot{\underline{r}}_i$$

$$P_s = \sum_i m_i \dot{\underline{r}}_i \frac{\partial \underline{r}_i}{\partial q_i} = \sum_i m_i \dot{\underline{r}}_i \partial \underline{r}_i / \partial q_s$$

(from the proof of Lagrange's eqns)

Therefore,

$$\sum_{s=1}^{n} \dot{q}_s P_s = \sum_s \sum_i m_i \dot{\underline{r}}_i \cdot \frac{\partial \underline{r}_i}{\partial q_s} \dot{q}_s = \sum_i m_i \dot{\underline{r}}_i \cdot \sum_s \dot{q}_s \frac{\partial \underline{r}_i}{\partial q_s} \qquad (12)$$

But

$$\dot{\underline{r}}_i = \sum_{s=1}^{n} \frac{\partial \underline{r}_i}{\partial q_s} \dot{q}_s + \frac{\partial \underline{r}_i}{\partial t} \qquad (13)$$

("but $\underline{r}_i$ does not depend explicitly on the time")

$\therefore$ (12), (13) implies

$$\sum_s P_s \dot{q}_s = \sum_i m_i \dot{\underline{r}}_i \cdot \dot{\underline{r}}_i = 2T$$

$$\therefore H = -L + \sum_{s=1}^{n} P_s \dot{q}_s = -T + V + 2T = V + T = E = \text{const}$$

Conclude: Since the system is conserved in energy.

Subject to the Condition that H does not depend explicitly on the time i.e the system does not contain moving constraints

A. Mech.

Note (2) $\{q_i, H\} = \sum_i \left(\frac{\partial q_j}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial q_j}{\partial p_i}\frac{\partial H}{\partial q_i} \right) =$

($\frac{\partial q_j}{\partial q_i} = \delta_{ij}$, $\frac{\partial q_j}{\partial p_i} = 0$)

$\frac{\partial H}{\partial p_j} = \dot{q}_j$. Also.

$\{p_j, H\} = \sum_i \left(\frac{\partial p_j}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial p_j}{\partial p_i}\frac{\partial H}{\partial q_i} \right) = -\frac{\partial H}{\partial q_j} = \dot{p}_j$

($\frac{\partial p_j}{\partial q_i} = 0$, $\frac{\partial p_j}{\partial p_i} = \delta_{ij}$)

Consequently Hamilton's eq'ns takes the symmetric form

$$\dot{q}_j = \{q_j, H\} \quad , \quad \dot{p}_j = \{p_j, H\} \qquad (14)$$

Note (3) If f is any function of t, q_i, p_i $(i = 1, 2, \ldots n)$

then $\frac{df}{dt} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p_i}\dot{p}_i \right) + \frac{\partial f}{\partial t} =$ (from Hamilton's eq'ns)

$\sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i} \right) + \frac{\partial f}{\partial t}$ *

$\therefore \frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t} \qquad (15)$

* which has it equivalent in quantum mechanics.

$\hat{f}_H = e^{\frac{i\hat{H}t}{\hbar}} \hat{f}_S e^{-\frac{i\hat{H}t}{\hbar}}$

$\hat{f}_H$ — Heisenberg Picture; $\hat{f}_S$ — Schrödinger Picture

$\frac{d\hat{f}_H}{dt} = \frac{i\hat{H}}{\hbar}\hat{f}_H - \frac{i}{\hbar}\hat{f}_H\hat{H} + \left[\frac{\partial \hat{f}}{\partial t}\right]_H =$

$\frac{1}{i\hbar}[\hat{f}_H, \hat{H}]$

This eqtn is identical to the eqtn in

$$\frac{d\hat{f}_H}{dt} = \frac{1}{i\hbar}[\hat{f}_H, H] + \left(\frac{\partial \hat{f}}{\partial t}\right)_H$$

Commutator

where $\hat{f}_H = e^{\frac{iHt}{\hbar}} \hat{f}_s e^{-\frac{iHt}{\hbar}}$ in Quantum Mechanics.

Heisenberg → Schrodinger

From eqtn (15) it is clear that f is a Constant of the motion

$\frac{df}{dt} = 0$ if $\{f, H\} = -\frac{\partial f}{\partial t}$ and if

$\{f, H\} = 0$ when f does not depend explicitely on the time.

The same result can be obtained in Quantum Mechanics.

* Lagrange's and Hamilton's Equation.

function for an Electromagnetic Field.

For a holonomic system Lagrange's eqtns take the form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_s} - \frac{\partial T}{\partial q_s} = Q_s \quad [I]0, \quad s = 1, 2, \ldots n$$

Q_s is the generalized Components of Forces.

Also if the system is Conservative ; then eqtn (2) may be expressed as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = 0 \quad [II](2), \quad s = 1, 2, \ldots n$$

$$L = T - V \; ; \; Q_s = -\frac{\partial V}{\partial q_s}$$

In fact eqtn (2) Can still be valid if the generalized Components of force are expressed in the form

$$Q_s = -\frac{\partial V}{\partial q_s} + \frac{d}{dt}\frac{\partial V}{\partial \dot{q}_s} \quad \text{and} \quad L = T - U(q_s, \dot{q}_s, t)$$

Here U is Called a generalized Potential function [II](3)

A good example of the latter Case is the electromagnetic field

* In order to show this we start by Maxwell's eqtns.

① $\nabla \times \underline{E} = -\frac{1}{c}\frac{\partial \underline{B}}{\partial t}$ $\quad c$ = velocity of light

$\underline{E}$ = electric field intensity ; $\underline{B}$ = magnetic induction

M.K.S or SI $\frac{1}{c}$ reduces to unity.

② $\nabla \times \underline{H} = \frac{4\pi}{c}\underline{J} + \frac{1}{c}\frac{\partial \underline{D}}{\partial t}$

in Case of M.K.S $\frac{4\pi}{c}$, $\frac{1}{c}$ reduces to unity respectively

$\underline{H}$ = Magnetic field intensity ; $\underline{J}$ = Conduction Current

$\underline{D}$ = displacement vector

③ $\nabla \cdot \underline{D} = 4\pi\rho$, ρ = Charge density

in MKS

④ $\nabla \cdot \underline{B} = 0$; $\underline{D} = \epsilon \underline{E}$ (ϵ : permittivity)

$\underline{H} = \mu \underline{B}$ (μ : permeability)

Also the Lorentz force is due to the E.M field.
q (+ve) i.e. electron $\Rightarrow -e$

$$\underline{F} = q\left(\underline{E} + \frac{1}{c}\,\underline{v} \times \underline{B}\right) \qquad (5)$$

Note: In the case when $\underline{B} = 0 \Rightarrow$ From (1)

$\nabla \times \underline{E} = 0 \Rightarrow \underline{E} = -\text{grad}\,\varphi$

$\underline{F} = q\underline{E} = -\text{grad}\,V$, $V = q\varphi$

and the field is Conservative.

The above case is not applied in the general case.
(In general case, we have from eq^{n} (4)
that $\underline{B} = \text{Curl}\,\underline{A}$ Since div Curl $= 0$ (6)
$\underline{A}$ is called the magnetic vector potential.
Substituting in eq^{n} (1) we obtain

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} = -\frac{\partial (\nabla \times \underline{A})}{\partial t} = -\nabla \times \frac{\partial \underline{A}}{\partial t} \Rightarrow$$

$$\nabla \times \left(\underline{E} + \frac{\partial \underline{A}}{\partial t}\right) = 0 \quad \text{which implies that}$$

$$\underline{E} + \frac{\partial \underline{A}}{\partial t} = -\text{grad}\,\varphi \qquad (7)$$

Since Curl grad $= 0$

$$\therefore \quad \underline{E} = -\frac{\partial \underline{A}}{\partial t} - \nabla\varphi$$

Subst in eq^{n} (5), ("velocity")

gives $\underline{F} = q\left[-\frac{\partial \underline{A}}{\partial t} - \nabla\varphi + \underline{v} \times (\nabla \times \underline{A})\right]$ (8)

But $[v \times (\nabla \times \underline{A})]_x = v_y (\nabla \times \underline{A})_z - v_z (\nabla \times \underline{A})_y =$

$v_y \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - v_z \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) \Rightarrow$

$$\{\underline{v}\times(\nabla\times\underline{A})\}_x = v_y\frac{\partial A_y}{\partial x} + v_z\frac{\partial A_z}{\partial x} - v_y\frac{\partial A_x}{\partial y} - v_z\frac{\partial A_x}{\partial z}$$

$$+ v_x\frac{\partial A_x}{\partial x} - v_x\frac{\partial A_x}{\partial x} =$$

$$\underline{v}\cdot\frac{\partial \underline{A}}{\partial x} - \underline{v}\cdot\left(\frac{\partial A_x}{\partial x}, \frac{\partial A_y}{\partial x}, \frac{\partial A_z}{\partial x}\right) \quad (9)$$

Since $A(x,y,z,t)$

$$\frac{dA}{dt} = \frac{\partial A}{\partial x}v_x + \frac{\partial A}{\partial y}v_y + \frac{\partial A}{\partial z}v_z + \frac{\partial A}{\partial t} \Rightarrow$$

$$\{\underline{v}\times(\nabla\times\underline{A})\}_x = \underline{v}\cdot\frac{\partial \underline{A}}{\partial x} - \frac{dA_x}{dt} + \frac{\partial A_x}{\partial t} \quad (11)$$

Subst. from (11) in (8) we find

$$F_x = q\left[-\frac{\partial \varphi}{\partial x} - \frac{\partial A_x}{\partial t} + \underline{v}\cdot\frac{\partial \underline{A}}{\partial x} - \frac{dA_x}{dt} + \frac{\partial A_x}{\partial t}\right] =$$

$$\left(\frac{\partial}{\partial x}(\underline{v}\cdot\underline{A})\right)$$

$$F_x = q\left[-\frac{\partial}{\partial x}(\varphi - \underline{v}\cdot\underline{A})\right] = Q_x$$

$$Q_s = -\frac{\partial U}{\partial q_s} + \frac{d}{dt}\frac{\partial U}{\partial \dot{q}_s}$$

$$-\frac{d}{dt}\left(\frac{\partial}{\partial v_x}\underline{v}\cdot\underline{A}(x,y,z,t)\right) \Rightarrow$$

$$F_x = q\left[-\frac{\partial}{\partial x}(\varphi - \underline{v}\cdot\underline{A}) + \frac{d}{dt}\frac{\partial}{\partial v_x}(\varphi - \underline{v}\cdot\underline{A})\right]$$

$\because$ as $\varphi(x,y,z,t) \Rightarrow \frac{\partial \varphi}{\partial v_x} = 0$

$$\therefore F_x = Q_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial v_x} \quad (12)$$ which is in the shape of (3)

$\frac{\partial}{\partial q_s}$ $\frac{\partial}{\partial \dot{q}_s}$

$$U = q\varphi - q\underline{v}\cdot\underline{A} \qquad (13)$$

Equation (12) is in the same form as eq^{n} (III)

The Lagrangian function L will thus be defined by

$$L = T - U = \frac{1}{2}mv^2 - q\varphi + q\underline{v}\cdot\underline{A} \qquad (14)$$

[m (mass of the charged particle)]

* The generalized components of momentum will be defined in this case by

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \quad \therefore \quad p_x = \frac{\partial L}{\partial v_x} = mv_x + qA_x$$

Therefore the momentum $\underline{p} = m\underline{v} + q\underline{A}$

$\underline{p}$ is the generalized momentum

$\underline{v}$ Velocity $\underline{A}$ vector potential (15)

The Hamiltonian function

$$H = -L + \sum_s p_s \dot{q}_s = -L + p_x v_x + p_y v_y + p_z v_z =$$

$$-L + v_x(mv_x + qA_x) + \text{---} + \text{---}$$

$$= -L + mv^2 + q\underline{A}\cdot\underline{v} \Rightarrow \text{from (14)}$$

$$\frac{1}{2}mv^2 + q\varphi \; * \Rightarrow \text{from (15)}$$

$$\Rightarrow H = \frac{1}{2m}|\underline{p} - q\underline{A}|^2 + q\varphi \qquad (16)$$

K.Imp